Exercises

1. Prove that the 1-manifold $(0, 1)$ is not compact.
   \[ \left\{ \frac{1}{n}, 1 \right\} : n \in \mathbb{Z}_+ \] is an open covering of $(0, 1)$ that does not admit any finite subcover.

2. Let $X \subset \mathbb{R}^2$ be the ”figure eight” represented as the union of the two unit circles tangent to the $x$-axis at the origin. Explain why $X$ is not a manifold, then prove it.
   $X$ fails to be locally Euclidean at the point $(0, 0)$. Indeed, suppose $U$ is a neighborhood of $(0, 0)$ in $X$ for which there exists a homeomorphism $f : U \rightarrow \mathbb{R}$ onto an open subset of $\mathbb{R}$. By restricting $f$ to a connected subset of $U$, we may assume $U$ is connected and contains neither $(0, 1)$ nor $(0, -1)$. Then $f$ maps $U$ to some interval $(a, b)$. The set $U' = U \setminus \{(0, 0)\}$ maps to $(a, c) \cup (c, b)$ where $c = f(0, 0)$. Note that $U'$ is partitioned by the quadrants into subsets $V_i, i = 1, \ldots, 4$ that are open relative to $U'$. Since $(a, c)$ is connected, at most one of the (open) sets $f(V_i)$ can intersect $(a, c)$. Similarly, at most one can intersect $(c, b)$. This is a contradiction, since each set $f(V_i)$ intersects either $(a, c)$ or $(c, b)$.

3. Prove that $[0, \varepsilon)$ is not homeomorphic to an open interval.
   If $f : [0, \varepsilon) \rightarrow (a, b)$ is a homeomorphism, then so is its restriction to $(0, \varepsilon)$. But the domain is connected and the codomain, $(a, f(0)) \cup (f(0), b)$, is not.

4. (**) Is the half-disk $H = \{ x^2 + y^2 < \varepsilon, y \geq 0 \}$ homeomorphic to an open disk?
   Let $D = \{ x^2 + y^2 < 1 \}$ and suppose $f : D \rightarrow H$ is a homeomorphism. Then so is the restriction $f : D' \rightarrow H'$ where $D' = D \setminus \{(0, 0)\}$ and $H' = H \setminus \{f(0, 0)\}$. But $H'$ is contractible, being homeomorphic to a product of an open interval and a closed interval, whereas $D'$ has the same homotopy type as $S^1$ and is therefore not contractible.

5. Let $f_1(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases}$. Is $f$ differentiable? $C^1$? twice differentiable?
   Yes. Yes. No.

6. Let $f_2(x) = \begin{cases} 0 & x \leq 0 \\ x^2 \sin \frac{1}{x} & x > 0 \end{cases}$. Is $f$ differentiable? $C^1$? twice differentiable?
   Yes. No. No.

7. Let $f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$. Show that $f$ is $C^\infty$. Is it real-analytic?
   It is clear that $f$ is real-analytic at any $x \neq 0$. By induction, for any $x \neq 0$,
   \[ f^n(x) = \begin{cases} 0 & x < 0 \\ R_n(x)e^{-\frac{1}{x}} & x > 0 \end{cases} \]
   where $R_n(x)$ are the rational functions defined by recurrence relation
   \[ R_{n+1}(x) = R'_n(x) + x^{-2}R_n(x), \quad R_0(x) = 1. \]
It remains to show that for all $n$

$$\lim_{x \to 0^+} R_n(x)e^{-\frac{1}{x}} = 0$$  \hspace{1cm} (1)$$

for then $f^{(n)}(0) = 0$ for all $n$, so that $f$ is $C^\infty$ and not real-analytic because its Taylor series vanishes in a neighborhood of 0 and $f$ does not. (1) reduces to showing

$$\lim_{x \to 0^+} x^d e^{-\frac{1}{x}} = 0$$

for any integer $d$; or equivalently,

$$\lim_{x \to 0^+} \frac{dx \log x - 1}{x} = -\infty,$$

which is clear, since $x \log x \to 0$ as $x \to 0^+$, by L’Hopital’s rule.

**Review from topology**

1. **Prove that a composition of continuous functions is continuous.**

   Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. For any set $V \subset Z$, we have

   $$(g \circ f)^{-1} V = f^{-1}(g^{-1} V).$$

   If $V$ is open in $Z$, then $g^{-1} V$ is open in $Y$, so that, by the above, $(g \circ f)^{-1} V$ is open in $X$. Hence, $g \circ f$ is continuous.

2. **Prove that the continuous image of a compact set is compact.**

   Let $f : X \to Y$ be continuous and $A$ a compact subset of $X$. We prove $f(A)$ is a compact subset of $Y$. Suppose $\{V_\alpha\}_{\alpha \in J}$ is an open cover of $f(A)$. Then $\{f^{-1} V_\alpha\}_{\alpha \in J}$ is an open cover of $A$. Hence, there exists a finite subset $F \subset J$ such that $\{f^{-1} V_\alpha\}_{\alpha \in F}$ covers $A$. It remains to check that $\{V_\alpha\}_{\alpha \in F}$ covers $f(A)$. Indeed, given $a \in A$, there is an element $\alpha \in F$ such that $a \in f^{-1} V_\alpha$. Thus, $f(a) \in V_\alpha$.

3. **Prove that the continuous image of a connected set is connected.**

   Let $f : X \to Y$ be continuous and $A$ a connected subset of $X$. We prove $f(A)$ is a connected subset of $Y$, by contradiction. Suppose $f(A) = V_1 \cup V_2$ is a separation of $f(A)$ into open subsets (relative to $f(A)$). Then $f^{-1} V_1$ and $f^{-1} V_2$ are disjoint open subsets of $X$ whose union contains $A$. Their intersection with $A$ defines a separation of $A$, which does not exist since $A$ is connected. Hence, $f(A)$ is connected.

4. **Show that a continuous bijection need not be a homeomorphism.**

   (Hint: Consider the mapping $\theta \in [0, 2\pi) \mapsto (\cos \theta, \sin \theta).$)

   The map $f : [0, 2\pi) \to S^1$ defined in the hint is a continuous bijection. It is not a homeomorphism since $S^1$ is compact and $[0, 2\pi)$ is not. Alternatively, one can argue directly that the inverse fails to be continuous at $(1, 0)$ by noting that there is a sequence in $S^1$ that converges to $(1, 0)$ but whose image in $[0, 2\pi)$ does not converge.
5. (*) Let $f : X \to Y$ be a continuous map between a compact space $X$ and a Hausdorff space $Y$. Show that $f$ is a closed map, i.e. it maps closed sets to closed sets.

Let $A$ be a closed set in $X$. Since a closed subset of a compact space is compact, $A$ is compact. Since a continuous image of a compact set is compact, $f(A)$ is compact. Since a compact subset of a Hausdorff space is closed, $f(A)$ is closed. Thus, $f$ maps closed sets to closed sets.

6. Let $f : S^1 \to \mathbb{R}^2$ be a continuous injection, a.k.a. a knot. Show that $f$ is a homeomorphism onto its image. (Remark: such a map is called an embedding.)

Let $K = f(S^1)$. The restriction $f : S^1 \to K$ is a continuous bijection. (As is customary, we denote the restriction of $f$ by the same name and let the context determine which function is meant.) It remains to show that $f^{-1}$ is continuous. By the closed set formulation of continuity, this is equivalent to showing that $f$ is a closed map. To complete the proof, we apply the result of the previous exercise.