The purpose of this exercise is to understand the theorem about partial fractions. This is really a theorem in algebra. It says that every algebraic fraction (quotient of polynomials) is equal to a sum of special expressions. The theorem appears in calculus because each of the special expressions can be integrated, so every algebraic fraction is a sum of integrable expressions. That means every algebraic fraction can be integrated.

Here’s the theorem. A polynomial is an expression of the form

\[p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.\]

The degree of \( p \) is \( \deg(p) = n \). Suppose you have an algebraic fraction \( \frac{p}{q} \).

- If \( \deg(p) \geq \deg(q) \), then \( \frac{p}{q} = r + \frac{s}{q} \), where \( r \) and \( s \) are polynomials and \( \deg(s) < \deg(q) \). For example:

\[
\frac{x^4 + x^3 - 3x + 2}{x^2 - x - 1} = x^2 + 2x + 3 + \frac{2x + 5}{x^2 - x - 1}
\]

- Every polynomial can be factored into a product of powers of distinct linear and irreducible (over the reals) quadratic factors. If the polynomial is monic (highest coefficient equals 1), then the factors can be monic too. For example:

\[
x^{10} - 2x^9 - 4x^8 + 4x^7 + 4x^6 + 18x^5 - 7x^4 - 4x^3 - 32x^2 - 16x - 16
\]

\[
= (x^2 + x + 1)^2 (x^2 + 1) (x - 2)^3 (x + 2)
\]

Finding the factors is the only part of computing the partial fraction decomposition that is not trivial.

- Every irreducible quadratic can be expressed as \((x + a)^2 + b^2\). For example, the factorization above becomes:

\[
x^{10} - 2x^9 - 4x^8 + 4x^7 + 4x^6 + 18x^5 - 7x^4 - 4x^3 - 32x^2 - 16x - 16
\]

\[
= ((x + .5)^2 + 0.75) (x^2 + 1) (x - 2)^3 (x + 2)
\]

- An algebraic fraction \( \frac{s}{q} \), where \( \deg(s) < \deg(q) \), can be written as a sum of fractions where the denominators are the factors of \( q \) and the numerators are either constants (for powers of linear factors) or linear polynomials (for powers for quadratic factors). Here is an example:

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Date: February 1, 2010.
\[
\frac{2x^9 - 19x^8 + 42x^7 + 52x^6 + 37x^5 - 57x^4 - 203x^3 - 95x^2 - 85x - 22}{x^{10} - 2x^9 - 4x^8 + 4x^7 + 4x^6 + 18x^5 - 7x^4 - 4x^3 - 32x^2 - 16x - 16}
\]

\[
= \frac{2x^9 - 19x^8 + 42x^7 + 52x^6 + 37x^5 - 57x^4 - 203x^3 - 95x^2 - 85x - 22}{((x + .5)^2 + 0.75)^2 (x^2 + 1)(x - 2)^3(x + 2)}
\]

\[
= \frac{3x + 1}{((x + .5)^2 + 0.75)^2} + \frac{2x + 1}{(x + .5)^2 + 0.75} - \frac{-x + 5}{x^2 + 1} + \frac{3}{(x - 2)^3} + \frac{5}{(x - 2)^2} - \frac{4}{x - 2} + \frac{3}{x + 2}
\]

- Another way to express the previous point is this. Suppose \( \frac{s}{q} \) is a quotient of polynomials with \( \deg(s) < \deg(q) \).
  - If \((x + a)^2 + b^2\)^m divides \(q\), then the partial fraction expansion of \( \frac{s}{q} \)
    includes a term of the form \( \frac{cx + d}{((x + a)^2 + b^2)^m} \).
  - If \((x + a)^m\) divides \(q\), then the partial fraction expansion of \( \frac{s}{q} \)
    includes a term of the form \( \frac{c}{(x + a)^m} \).

At this point, you may be saying to yourself: “This is a mess! How can I be expected to calculate such things?” You are right. The calculations, except for the factorization part, are trivial but very long and prone to arithmetic error. That’s why we have computers. \(^{1}\) Later you will learn computer methods for calculating partial fractions. All you have to learn now is the expected form of a partial fraction decomposition. If you are given an algebraic fraction with a factored denominator, you should be able to predict the form of partial fraction decomposition. For example, if you are asked for the form of the partial fraction decomposition of:

\[
\frac{x^3 - 3x^2 + 4x - 5}{(x^2 + x + 1)(x + 2)^2}
\]

you would respond:

\[
\frac{ax + b}{x^2 + x + 1} + \frac{c}{(x + 2)^2} + \frac{d}{x + 2}
\]

Find the form of the partial fraction decomposition for the following expressions:

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\(^{1}\) All the example calculations were done with Maxima. And how was I so clever that I picked an example that resulted in partial fractions with integer instead of decimal or fractional coefficients? I started with the final partial fraction decomposition and worked backwards. If you start with a random quotient of polynomials, you can expect the partial fraction decomposition to have really messy coefficients.
Essay question due February 15. A partial fraction decomposition has two kinds of terms: \( \frac{a}{x + b} \) and \( \frac{ax + b}{((x + c)^2 + d^2)^m} \). Partial fraction decomposition is useful because all of these forms can be integrated. Can you integrate them? You might start by substituting numbers for \( a, b, c \) and \( d \) to simplify the problems.

(1) Find the antiderivative of \( \frac{a}{x + b} \).

(2) Find the antiderivative of \( \frac{a}{(x + b)^m} \).

(3) Find the antiderivative of \( \frac{ax + b}{(x + c)^2 + d^2} \). Hint: First solve the problem with \( c = 0 \), then use the substitution \( u = x + c \).

(4) EXTRA CREDIT. Find the antiderivative of \( \frac{ax + b}{((x + c)^2 + d^2)^2} \). Then describe a procedure for finding the anti-derivative of \( \frac{ax + b}{((x + c)^2 + d^2)^m} \).