Midterm II

Monday, August 5

2 hours

Name: ___________________________________

Instructions

1. This is closed book, closed notes exam.
2. No calculators of any kind are allowed.
3. Show all the calculations.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

Good Luck 😊
Notation for Mean-Variance Portfolios with \( n \) Risky Assets

Asset returns:
\[
\begin{bmatrix}
  r_1 \\
  r_2 \\
  \vdots \\
  r_n
\end{bmatrix}_{n \times 1}
\]

Mean vector of asset returns:
\[
\mu = \begin{bmatrix}
  \mu_1 \\
  \mu_2 \\
  \vdots \\
  \mu_n
\end{bmatrix}_{n \times 1}
\]

Covariance matrix of asset returns:
\[
\Sigma = \begin{bmatrix}
  \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1n} \\
  \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}^2
\end{bmatrix}_{n \times n}
\]

Portfolio weights (shares) on assets:
\[
\lambda = \begin{bmatrix}
  \lambda_1 \\
  \lambda_2 \\
  \vdots \\
  \lambda_n
\end{bmatrix}_{n \times 1}
\]

Vector of ones:
\[
l_n = \begin{bmatrix}
  1 \\
  1 \\
  \vdots \\
  1
\end{bmatrix}_{n \times 1}
\]

Risk-free return: \( r_f \)

Share of wealth invested in risk-free asset: \( \lambda_0 \)
1. (20 points). Consider a portfolio consisting of $n$ risky assets.
   
a. Using the matrix notation on page 1, what is the return on a portfolio with weights $\lambda$? Denote this return by $r_\lambda$.
   
   $$r_\lambda = \lambda' r$$
   
   b. Using the matrix notation on page 1, what is the mean return on a portfolio with weights $\lambda$? Denote this mean return by $\mu_\lambda$.
   
   $$\mu_\lambda = \lambda' \mu = \mu' \lambda$$
   
   c. Using the matrix notation on page 1, what is the variance of the return on a portfolio with weights $\lambda$? Denote this variance by $\sigma^2_{\lambda}$.
   
   $$\sigma^2_{\lambda} = \lambda' \Sigma \lambda$$
   
   d. Suppose that an investor spends a fraction $\lambda_0$ of his wealth on the risk-free asset, and the rest on risky assets. Let $\lambda$ be the portfolio weights on the risky assets. Using the matrix notation on page 1, write the utility maximization problem of an investor with mean-variance utility function $u(\mu, \sigma) = \mu - \frac{\gamma}{2} \sigma^2$.
   
   $$\max_{\lambda_0, \lambda} \lambda_0 r_f + \lambda' \mu - \frac{\gamma}{2} \lambda' \Sigma \lambda$$
   
   s.t.
   
   $$\lambda_0 + \lambda' 1_n = 1$$

You can substitute the budget constraint into the objective function, so the above simplifies to:

$$\max_{\lambda} \lambda' (\mu - 1_n r_f) - \frac{\gamma}{2} \lambda' \Sigma \lambda$$

or

$$\max_{\lambda} (\mu - 1_n r_f)' \lambda - \frac{\gamma}{2} \lambda' \Sigma \lambda$$
2. (20 points). In order to find optimal portfolios for mean-variance investors, we use the Optimization Toolbox function `quadprog.m`. This function solves problems of the form:

\[
\begin{align*}
\min & \quad 0.5x'\mathbf{H}x + f'x \\
x & \quad \text{subject to: } A\mathbf{x} \leq b \\
& \quad A_{eq}\mathbf{x} = b_{eq} \\
& \quad lb \leq \mathbf{x} \leq ub
\end{align*}
\]

The function is then used as follows:

\[
x = \text{quadprog}(\mathbf{H},f,A,b,A_{eq},b_{eq},lb,ub,x0,\text{options});
\]

a. How would you rewrite the optimal portfolio selection problem in the previous question, so that it maps into the form required by `quadprog.m`?

Since `quadprog.m` solves minimization problems, the portfolio selection problem can be written as:

\[
\min_{\lambda} \frac{\gamma}{2} \lambda'\Sigma\lambda - (\mu - 1_n r_f)'\lambda
\]

Finding the maximum of any function \( f(x) \) is the same as finding the minimum of \(-f(x)\).

b. How would you define the input \( \mathbf{H} \), if you wish to solve the optimal portfolio selection problem using `quadprog.m`.

\[
\mathbf{H} = \gamma'\Sigma
\]
c. How would you define the input $f$, if you wish to solve the optimal portfolio selection problem using `quadprog.m`.

$$ f = - (\mu - 1_n r_f) $$

d. A part of the program `CML.m` is given below.

```matlab
lb = -inf*(ones(size(x0))); ub = inf*ones(size(x0));
min_lambda_0 = 0;
A = ones(size(x0))'; b = 1-min_lambda_0;
x = quadprog(H,f,A,b,[],[],lb,ub,x0,options);
lambda_0 = 1-sum(x);
```

How would you modify the above code, for case when short selling of risky assets is not allowed?

Instead of

```matlab
lb = -inf*(ones(size(x0)));
```

we need to have

```matlab
lb = 0*(ones(size(x0)));
```
3. (15 points). The next figure describes the portfolio opportunity set for risky assets, in the mean-variance space.

a. Label the axis on the above graph.

b. On the above graph, indicate the **efficient frontier**.

c. Suppose that two investors have mean-variance utility functions

\[ u(\mu, \sigma) = \mu - \frac{\gamma_1}{2} \sigma^2 \]

and

\[ u(\mu, \sigma) = \mu - \frac{\gamma_2}{2} \sigma^2 \]

with \( \gamma_1 > \gamma_2 \). On the above graph, indicate two possible optimal portfolios that these investors might choose. Denote the portfolio of investor 1 by \( A_1 \) and the portfolio of investor 2 by \( A_2 \).

d. Suppose a risk-free asset with return \( r_f \), is available. On the above graph, draw the Capital Market Line (CML).

e. Consider the same two investors from section c, but now the risk-free asset is available. On the above graph, indicate two possible optimal portfolios that these investors might choose. Denote the portfolio of investor 1 by \( B_1 \) and the portfolio of investor 2 by \( B_2 \).

f. Circle the correct answer from the following:

i. When a risk-free asset is available, all mean-variance investors will always hold the same portfolio.

ii. When a risk-free asset is available, all mean-variance investors will always hold the same portfolio of risky assets.
The efficient frontier is the thick curve, on the top of opportunity set.
4. (35 points). Suppose that investors can invest in a risk-free asset with return \( r_f \), and two risky assets, with random returns \( r_1, r_2 \) mean and covariance of returns:

\[
\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}
\]

Investors have mean-variance utility function \( u(\mu, \sigma) = \mu - \frac{\gamma}{2} \sigma^2 \).

a. Find the return, the mean return and variance of the return on portfolio \( p \) with weights \( \lambda_0, \lambda_1, \lambda_2 \). That is, calculate \( r_p, \mu_p, \sigma_p^2 \).

\[
\begin{align*}
r_p &= \lambda_0 r_f + \lambda_1 r_1 + \lambda_2 r_2 \\
\mu_p &= \lambda_0 \mu_f + \lambda_1 \mu_1 + \lambda_2 \mu_2 \\
\sigma_p^2 &= \lambda_1^2 \sigma_1^2 + \lambda_2^2 \sigma_2^2 + 2 \lambda_1 \lambda_2 \sigma_{12}
\end{align*}
\]

b. Write the problem of an active investor, who wants to find his optimal portfolio. Substitute the budget constraint into the objective function.

\[
\max_{\lambda_0, \lambda_1, \lambda_2} \lambda_0 r_f + \lambda_1 \mu_1 + \lambda_2 \mu_2 - \frac{\gamma}{2} \left[ \lambda_1^2 \sigma_1^2 + \lambda_2^2 \sigma_2^2 + 2 \lambda_1 \lambda_2 \sigma_{12} \right]
\]

s.t.

\[
[BC]: \quad \lambda_0 + \lambda_1 + \lambda_2 = 1
\]

After substituting the budget constraint into the objective function, the problem simplifies to:

\[
\max_{\lambda_1, \lambda_2} \lambda_1 (\mu_1 - r_f) + \lambda_2 (\mu_2 - r_f) - \frac{\gamma}{2} \left[ \lambda_1^2 \sigma_1^2 + \lambda_2^2 \sigma_2^2 + 2 \lambda_1 \lambda_2 \sigma_{12} \right]
\]
c. Write the first order conditions for optimal weights $\lambda_1, \lambda_2$. No need to solve for the optimal weights.

$$\left[\lambda_1\right]: \mu_1 - r_f - \gamma [\lambda_1 \sigma_1^2 + \lambda_2 \sigma_{12}] = 0$$

$$\left[\lambda_2\right]: \mu_2 - r_f - \gamma [\lambda_2 \sigma_2^2 + \lambda_1 \sigma_{12}] = 0$$

d. We solved analytically for the optimal portfolio in class, and the weights on the two risky assets are:

$$\lambda_1^{opt} = \frac{1}{\gamma} \left[ \frac{\mu_1 - r_f - (\sigma_{12} / \sigma_2^2)(\mu_2 - r_f)}{\sigma_1^2 - (\sigma_{12} / \sigma_2^2)^2} \right]$$

$$\lambda_2^{opt} = \frac{1}{\gamma} \left[ \frac{\mu_2 - r_f - (\sigma_{12} / \sigma_1^2)(\mu_1 - r_f)}{\sigma_2^2 - (\sigma_{12} / \sigma_1^2)^2} \right]$$

You can use this solution to solve the problem in this section.

Suppose the financial market consists of two investors, with different beliefs about the assets’ mean returns:

$$\mu^1 = \begin{bmatrix} 5\% \\ 3\% \end{bmatrix}, \quad \mu^2 = \begin{bmatrix} 3\% \\ 5\% \end{bmatrix}$$

Both investors have the same beliefs about the covariance of returns:

$$\Sigma = \begin{bmatrix} 4\% & 0\% \\ 0\% & 4\% \end{bmatrix}$$

The variance aversion parameters of the two investors are $\gamma_1$ and $\gamma_2$. Find the optimal portfolios and the tangent portfolios of the two investors when the risk-free rate is $r_f = 2\%$. Write your results in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Weights</th>
<th>Investor 1</th>
<th>Investor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal portfolio</td>
<td>$\lambda_1^{opt}$</td>
<td>$\frac{0.75}{\gamma_1}$</td>
<td>$\frac{0.25}{\gamma_2}$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2^{opt}$</td>
<td>$\frac{0.25}{\gamma_1}$</td>
<td>$\frac{0.75}{\gamma_2}$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_0^{opt}$</td>
<td>$1 - \frac{1}{\gamma_1}$</td>
<td>$1 - \frac{1}{\gamma_2}$</td>
</tr>
<tr>
<td>Tangent portfolio</td>
<td>$\lambda_1^T$</td>
<td>0.75</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2^T$</td>
<td>0.25</td>
<td>0.75</td>
</tr>
</tbody>
</table>
Since the two risky assets are uncorrelated, the optimal portfolios are calculated with the much simplified formulas:

\[
\lambda_1^{opt} = \frac{1}{\gamma} \left[ \frac{\mu_1 - r_f}{\sigma_1^2} \right] \\
\lambda_2^{opt} = \frac{1}{\gamma} \left[ \frac{\mu_2 - r_f}{\sigma_2^2} \right] \\
\lambda_0^{opt} = 1 - \lambda_1^{opt} - \lambda_2^{opt}
\]

The tangent portfolio is found by normalizing by \(1 - \lambda_0\):

\[
\lambda_1^T = \frac{\lambda_1^{opt}}{1 - \lambda_0}, \quad \lambda_2^T = \frac{\lambda_2^{opt}}{1 - \lambda_0}
\]
e. Suppose that both investors invest $w in risky assets (in their tangent portfolio). Find the market portfolio, i.e. find the equilibrium weights $\lambda_1^M, \lambda_2^M$.

Since both investors have equal size portfolio of risky assets, the market portfolio weights are averages of the individually chosen tangent portfolios:

$$
\lambda_1^M = \frac{1}{2} 0.75 + \frac{1}{2} 0.25 = 0.5
$$

$$
\lambda_2^M = \frac{1}{2} 0.25 + \frac{1}{2} 0.75 = 0.5
$$

Notice that you can calculate just one of the shares, say $\lambda_1^M$, and then the other share is obtained by $1 - \lambda_2^M$, because the shares must add up to 1.

f. Show that given the two investor’s beliefs, they will both think that their own portfolios have a positive Alpha. Find that Alpha.

The two SMLs for the investors are

$$
\mu^1 = r_f + \beta^1 (\mu_M^1 - r_f)
$$

$$
\mu^2 = r_f + \beta^2 (\mu_M^2 - r_f)
$$

Where $\mu^1$ and $\mu^2$ are the perceived expected returns on their own portfolios. Notice that they will both perceive the same expected return on the market portfolio, but this is a result of the specific numbers chosen for this question and not true in general:

$$
\mu_M^1 = 0.5 \cdot 5\% + 0.5 \cdot 3\% = 4\%
$$

$$
\mu_M^2 = 0.5 \cdot 3\% + 0.5 \cdot 5\% = 4\%
$$

They will also find the same Betas. The Beta on investor 1 portfolio:

$$
\beta^1 = \frac{\text{Cov}(0.75\hat{r}_1 + 0.25\hat{r}_2, 0.5\hat{r}_1 + 0.5\hat{r}_2)}{\text{Var}(0.5\hat{r}_1 + 0.5\hat{r}_2)}
$$

$$
= \frac{0.75 \cdot 0.5\sigma_1^2 + 0.25 \cdot 0.5\sigma_2^2}{0.5^2 \sigma_1^2 + 0.5^2 \sigma_2^2} = \frac{0.75 + 0.25}{0.5 + 0.5} = 1
$$
Since the assets are uncorrelated, the pairwise covariance terms are zero. The variances are the same for both assets, so they cancel out.

Similarly, the Beta on investor 2 portfolio:

\[
\beta^2 = \frac{\text{Cov}(0.25\bar{r}_1 + 0.75r_2, 0.5\bar{r}_1 + 0.5r_2)}{\text{Var}(0.5\bar{r}_1 + 0.5r_2)} \\
= \frac{0.25 \cdot 0.5\sigma^2_1 + 0.75 \cdot 0.5\sigma^2_2}{0.5^2\sigma^2_1 + 0.5^2\sigma^2_2} = \frac{0.25 + 0.75}{0.5 + 0.5} = 1
\]

Finally, the perceived expected returns on their portfolios, based on their own beliefs, are also the same:

\[
\mu^1 = 0.75 \cdot 5 + 0.25 \cdot 3 = \frac{15}{4} + \frac{3}{4} = \frac{18}{4} = 4.5\%
\]

\[
\mu^2 = 0.25 \cdot 3 + 0.75 \cdot 5 = \frac{3}{4} + \frac{15}{4} = \frac{18}{4} = 4.5\%
\]

The Alphas are therefore:

\[
\alpha^1 = \mu^1 - r_f - \beta^1(\mu^1_M - r_f) = 4.5 - 2 - 1 \cdot (4 - 2) = 0.5\%
\]

\[
\alpha^2 = \mu^2 - r_f - \beta^2(\mu^2_M - r_f) = 4.5 - 2 - 1 \cdot (4 - 2) = 0.5\%
\]
g. Suppose that the true mean returns are:

\[ \mu = \begin{bmatrix} 4 \% \\ 4 \% \end{bmatrix} \]

Show that the Alphas on the two investors’ portfolios, when we use the correct beliefs about expected returns, are both zero.

The mean return on the market portfolio, using correct beliefs, is:

\[ \mu_M = 0.5 \mu_1 + 0.5 \mu_2 = 0.5 \cdot 4\% + 0.5 \cdot 4\% = 4\% \]

The mean returns on the two investors’ portfolios, using correct beliefs:

\[ \mu^1 = 0.75 \cdot 4\% + 0.25 \cdot 4\% = 4\% \]
\[ \mu^2 = 0.25 \cdot 4\% + 0.75 \cdot 4\% = 4\% \]

The Betas are the same as before, because they do not depend on whether the beliefs are correct or not, but only on the chosen portfolios.

The Alphas are therefore:

\[ \alpha^1 = \mu^1 - r_f - \beta^1 (\mu_M - r_f) = 4 - 2 - 1 \cdot (4 - 2) = 0 \]
\[ \alpha^2 = \mu^2 - r_f - \beta^2 (\mu_M - r_f) = 4 - 2 - 1 \cdot (4 - 2) = 0 \]
5. (10 points). Suppose that Carlos (aka Danger) decided to be a passive investor, and divide his financial wealth between the risk-free asset and the market portfolio. Carlos cares only about the mean and variance of returns, and his utility function is \( u(\mu, \sigma) = \mu - \frac{\gamma}{2} \sigma^2 \). Let the fraction of his wealth invested in risk-free asset be \( \lambda_0 \), the return on the risk-free asset is \( r_f \), and the return on market portfolio is \( r_M \), with mean \( \mu_M \) and variance \( \sigma_M^2 \).

a. Write Carlos’ utility maximization problem, and find the optimal share invested in the market portfolio, \( 1 - \lambda_0 \).

Carlos’ portfolio has the following rate of return, mean and variance:

- Rate of return: \( r = \lambda_0 r_f + (1 - \lambda_0) r_M \)
- Mean: \( \mu = \lambda_0 r_f + (1 - \lambda_0) \mu_M \)
- Variance: \( \sigma^2 = (1 - \lambda_0)^2 \sigma_M^2 \)

Thus, his utility maximization problem is:

\[
\max_{\lambda_0} \lambda_0 r_f + (1 - \lambda_0) \mu_M - \frac{\gamma}{2} (1 - \lambda_0)^2 \sigma_M^2
\]

The first order condition for optimal \( \lambda_0 \) is:

\[
r_f - \mu_M + \gamma (1 - \lambda_0) \sigma_M^2 = 0
\]

Solving for \( 1 - \lambda_0^\text{opt} \):

\[
1 - \lambda_0^\text{opt} = \frac{\mu_M - r_f}{\gamma \sigma_M^2}
\]
b. Provide a brief economic intuition about the result you found in the last section. In particular, discuss how his investment decision depends on the excess return on the market portfolio, on the degree of his variance-aversion, and on the variance of the market portfolio.

For Carlos, the market portfolio is the only risky asset available. The higher is the excess return on the risky asset (higher $\mu_M - r_f$), the more will Carlos invest in it. The more risk-averse he is (higher $\gamma$) and the higher is the risk of the market portfolio (higher $\sigma_M^2$) the less will Carlos invest in the risky asset.