Midterm I

Wednesday, July 24

1 hour, 15 minutes

Name: ________________________________

Instructions

1. This is closed book, closed notes exam.
2. No calculators of any kind are allowed.
3. Show all the calculations.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

Good Luck 😊
1. (20 points). Consider the following lotteries (payoffs are in dollars):

\[ A = \begin{cases} 
1000 & \text{w.p. 4/5} \\
200 & \text{w.p. 1/5} 
\end{cases} \quad B = \begin{cases} 
1000 & \text{w.p. 2/5} \\
900 & \text{w.p. 2/5} \\
0 & \text{w.p. 1/5} 
\end{cases} \]

a. Prove that lottery \( A \) First Order Stochastically Dominates lottery \( B \).

The next table shows the values of the survival functions for both lotteries, for all possible payoffs \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 1 - F^A(x) )</th>
<th>comparison</th>
<th>( 1 - F^B(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \geq 1000 )</td>
<td>0</td>
<td>=</td>
<td>0</td>
</tr>
<tr>
<td>( 900 \leq x &lt; 1000 )</td>
<td>4/5</td>
<td>&gt;</td>
<td>2/5</td>
</tr>
<tr>
<td>( 200 \leq x &lt; 900 )</td>
<td>4/5</td>
<td>=</td>
<td>4/5</td>
</tr>
<tr>
<td>( 0 \leq x &lt; 200 )</td>
<td>1</td>
<td>&gt;</td>
<td>4/5</td>
</tr>
<tr>
<td>( x &lt; 0 )</td>
<td>1</td>
<td>=</td>
<td>1</td>
</tr>
</tbody>
</table>

We see that \( 1 - F^A(x) \geq 1 - F^B(x) \) for all \( x \), which is the definition of \( A \ FOSD \ B \).

b. Suppose that George’s preferences over lotteries are described by EUT (Expected Utility Theory), with increasing and convex vNM utility function \( u \). This means that George is risk-seeking. We can conclude that George prefers lottery \( B \) over lottery \( A \). True/False, circle the correct answer and briefly explain without proving anything (use results proved in class or in the course notes).

We proved that EUT satisfies First Order Stochastic Dominance, and that any investor with increasing vNM utility function, whether risk-averse, risk-seeking or risk-neutral, will choose the dominant lottery over the dominated.
c. Suppose that Kathy is **risk-averse**. Then, she is willing to pay at least $800 for lottery $B$. True/False, circle the correct answer, and explain briefly.

The mean payoff of lottery $B$ is:

\[ E(B) = \frac{2}{5} \cdot 1000 + \frac{2}{5} \cdot 900 + \frac{1}{5} \cdot 0 = 400 + 360 = 760 \]

If Kathy is risk averse, she is willing to pay less than 760 for lottery $B$.

d. Kathy’s certainty equivalent to lottery $A$ is $800. Find her risk premium for this lottery.

The mean payoff of lottery $A$ is:

\[ E(A) = \frac{4}{5} \cdot 1000 + \frac{1}{5} \cdot 200 = 840 \]

Thus, the risk premium is:

\[ RP = E(A) - CE = 840 - 800 = 40 \]

This is the amount she is willing to pay in order to receive the mean payoff of the lottery, and avoid playing the lottery itself.
2. (15 points). Define **risk-averse** individual in the following cases:
   a. General preferences.

An individual is risk-averse if he prefers the mean of any lottery $L \in \mathcal{L}$ over the lottery itself:

$$E[L] \succ L.$$ 

b. Preferences described by EUT (Expected Utility Theory).

For any lottery $L \in \mathcal{L}$,

$$u(E(x)) > E[u(x)]$$

c. Preferences described by MVT (Mean-Variance Theory).

For any lottery $L \in \mathcal{L}$, with mean $\mu$ and standard deviation $\sigma > 0$,

$$u(\mu, 0) > u(\mu, \sigma)$$
3. (10 points). Let \( u(x) \) be vNM utility function, such that \( u(0) = 10 \) and \( u(1000) = 20 \). Find another vNM utility function \( v(x) \), which represents the same preferences as \( u(x) \), and satisfies \( v(0) = 0 \) and \( v(1000) = 1 \).

We know that \( v(x) \) represents the same preferences as \( u(x) \) if \( v(x) = a + bu(x) \) for some number \( a \) and positive number \( b \). Thus, we need to solve for \( a \) and \( b \) from the next two equations:

\[
\begin{align*}
v(0) &= a + bu(0) \\
v(1000) &= a + bu(1000)
\end{align*}
\]

Or

\[
\begin{align*}
0 &= a + b \cdot 10 \\
1 &= a + b \cdot 20
\end{align*}
\]

From the first equation we have \( a = -10b \). Plugging into the second equation gives

\[
1 = -10b + 20b
\]

\( \Rightarrow b = 0.1, \ a = -1 \)

Thus, \( v(x) = -1 + 0.1u(x) \) represents the same preferences as \( u(x) \), and satisfies \( v(0) = 0 \) and \( v(1000) = 1 \).
4. (20 points). Suppose Richard has vNM utility function is \( u \), and that Richard is risk averse. His initial wealth is \( w \), and he can divide it between a risk-free asset with guaranteed return of \( r_f \) and a risky asset with random return \( r \). Let \( x \in [0,w] \) denote the amount invested in the risky asset.

a. Write Richard’s asset allocation problem.

His wealth next period is:
\[
(w - x)(1+r_f) + x(1+r) = w(1+r_f) - x(1+r_f) + x(1+r)
\]
\[
= w(1+r_f) + x(r-r_f)
\]

Optimal asset allocation problem:
\[
\max_{0 \leq x \leq w} E \left[ u \left( w(1+r_f) + x(r-r_f) \right) \right]
\]

b. Derive the first order necessary condition for optimal investment.

Denote the optimal investment in risky asset by \( x^* \).

\[
\frac{d}{dx} E[u(x)] = E \left[ u'(w(1+r_f) + x^*(r-r_f))(r-r_f) \right] = 0
\]

Where \( x^* \) denotes the optimal investment in risky asset.

c. Verify that the second order sufficient condition for optimum is satisfied.

\[
\frac{d^2}{dx^2} E[u(x)] = E \left[ u''(w(1+r_f) + x(r-r_f))(r-r_f)^2 \right] < 0
\]

The inequality follows from the risk aversion, \( u''(\cdot) < 0 \). Thus, since the objective function is concave, the first order condition gives a global maximum.
d. Prove that Richard will invest positive amount in the risky asset as long as the expected return on this asset is greater than the risk-free return.

Positive investment in the risky asset, \( x^* > 0 \), is equivalent to positive slope of the objective function at \( x = 0 \):

\[
E'\left[u'(w(1+r_f) + x(r-r_f))\right]_{x=0} > 0
\]
\[
E'\left[u'(w(1+r_f))\right] > 0
\]
\[
u'[w(1+r_f)]E(r-r_f) > 0
\]
\[
E(r-r_f) > 0
\]
\[
E(r) > r_f
\]

Notice that \( u'[w(1+r_f)] \) is non-random, and positive (due to monotonicity assumption of the vNM utility function). Thus, division by this term does not change the direction of the inequality.
5. (10 points). Suppose two individuals have vNM utility functions:
\[ u_1(x) = \frac{x^1 - \rho - 1}{1 - \rho}, \quad u_2(x) = \frac{x^{1 - \sigma} - 1}{1 - \sigma} \]

a. Suppose \( \rho = 4 \) and \( \sigma = 3 \). Which vNM utility function is more risk averse, \( u_1 \) or \( u_2 \)? Hint: The easiest way to answer this question is to calculate the Arrow-Pratt coefficient of relative risk aversion.

\[
\begin{align*}
    u_1'(x) &= x^{-\rho} \\
    u_1''(x) &= -\rho x^{-\rho - 1} \\
    RRA_1 &= -\frac{u_1''(x)}{u_1'(x)} = -\rho x^{-\rho - 1} \cdot x = \rho = 4
\end{align*}
\]

Thus, \( RRA_2 = \sigma = 3 \). With the given numbers, \( u_1 \) is more risk averse than \( u_2 \).

b. Based on your answer to the previous section, who will invest more in risky asset, \( u_1 \) or \( u_2 \)? No need to prove anything. In your answer refer to a result proved in class or in the notes.

We proved in the notes that investment in risky asset is decreasing in the degree of risk aversion. Since we found that \( u_1 \) is more risk averse than \( u_2 \), we conclude that \( u_2 \) will invest more than \( u_1 \) in any risky asset.
6. (10 points). Suppose that the vNM utility function $u$ is quadratic, i.e.
$u(x) = x - bx^2$, $b > 0$. Prove that there exists a mean-variance utility function $v(\mu, \sigma)$, which describes the same preferences as $u$.

$$E[u(x)] = E[x - bx^2] = E(x) - bE(x^2)$$
$$= \mu - b(\sigma^2 + \mu^2) \equiv v(\mu, \sigma)$$

Here we used the fact that $Var(x) = E(x^2) - E(x)^2$.

7. (5 points). Consider the Prospect Theory value function in the next graph:

The above value function is (circle the correct answer):

i. Risk-seeking for gains and risk-averse of losses,
ii. Risk-seeking for gains and losses,
iii. Risk-averse for gains and risk-seeking for losses,
iv. Risk-averse for gains and losses.
8. (10 points). Consider a lottery:

\[
L = \begin{cases} 
1000 & \text{w.p. } 2/5 \\
900 & \text{w.p. } 2/5 \\
0 & \text{w.p. } 1/5
\end{cases}
\]

Suppose that the prospect theory value function is \( v(x) \), and probability weighting function is \( w(p) \).

a. Write the original Prospect Theory utility from this lottery, \( PT(L) \).

\[
PT(L) = w\left(\frac{1}{5}\right)v(0) + w\left(\frac{2}{5}\right)v(900) + w\left(\frac{2}{5}\right)v(1000)
\]

The general formula used above is:

\[
PT(L) = \sum_{i=1}^{n} w(p_i)v(x_i)
\]

b. Write the Cumulative Prospect Theory utility from this lottery, \( CPT(L) \).

We order the payoffs in increasing order: \( 0 < 900 < 1000 \). The corresponding cumulative probabilities are:

\[
F_1 = \frac{1}{5}, \quad F_2 = \frac{3}{5}, \quad F_3 = 1
\]

As always, we define \( F_0 = 0 \). Thus,

\[
CPT(L) = \left[ w\left(\frac{1}{5}\right) - w(0) \right]v(0) + \left[ w\left(\frac{3}{5}\right) - w\left(\frac{1}{5}\right) \right]v(900) + \left[ w(1) - w\left(\frac{3}{5}\right) \right]v(1000)
\]

The general formula used above is:

\[
CPT(L) = \sum_{i=1}^{n} [w(F_i) - w(F_{i-1})]v(x_i)
\]