Midterm I

Wednesday, July 18

1 hour, 15 minutes

Name: ___________________________________

Instructions

1. This is closed book, closed notes exam.
2. No calculators of any kind are allowed.
3. Show all the calculations.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

Good Luck 😊
1. (10 points). Consider the following lotteries:

\[
A = \begin{cases} 
1000 & \text{w.p. } 3/4 \\
400 & \text{w.p. } 1/4 
\end{cases} \quad B = \begin{cases} 
1000 & \text{w.p. } 1/4 \\
800 & \text{w.p. } 1/4 \\
0 & \text{w.p. } 1/2 
\end{cases}
\]

a. Describe the payoffs and corresponding probabilities of the compound lottery \( C = 0.5A + 0.5B \).

\[
C = \begin{cases} 
1000 & \text{w.p. } 4/8 \\
800 & \text{w.p. } 1/8 \\
400 & \text{w.p. } 1/8 \\
0 & \text{w.p. } 2/8 
\end{cases}
\]

b. Which statement describes the relationship between lotteries \( A \) and \( B \)? Circle the correct answer):

- A First Order Stochastically Dominates \( B \).
- \( B \) First Order Stochastically Dominates \( A \).
- Neither lottery dominates the other.
- In order to determine First Order Stochastic Dominance, we need to know the vNM utility function.
2. (10 points). Let $A$ and $B$ be two continuous lotteries, with pdfs $f_A(x)$ and $f_B(x)$, and let $U$ be expected utility functional. Prove that

$$U(\lambda A + (1 - \lambda)B) = \lambda U(A) + (1 - \lambda)U(B)$$

for any $\lambda \in [0,1]$.

$$U(\lambda A + (1 - \lambda)B) = \int_{-\infty}^{\infty} u(x)[\lambda f_A(x) + (1 - \lambda) f_B(x)]dx$$

$$= \lambda \int_{-\infty}^{\infty} u(x) f_A(x) dx + (1 - \lambda) \int_{-\infty}^{\infty} u(x) f_B(x) dx$$

$$= \lambda U(A) + (1 - \lambda)U(B)$$

3. (10 points). Define **risk-averse** individual.

An individual is risk-averse if he prefers the mean of any lottery $L \in \mathcal{L}$ over the lottery itself: $E[L] \succ L$.

Remark: if preferences can be represented by EUT, then the definition can be written as $u[E(x)] > E[u(x)]$

Since EUT was not given in the question, your answer cannot use it.

4. (10 points). If individual is risk-averse, and his preferences are represented with expected utility functional, then his vNM utility must be **concave** convex (circle the correct answer).
5. (10 points). Let \( u(x) = \ln(x) \) be vNM utility function, and let
\( v(x) = -13 + 7 \cdot \ln(x) \) be another vNM utility function. The utility function \( v \)
describes the same preferences as \( u \). True/false, circle the correct answer and
briefly explain your choice.

We proved that vNM is invariant under increasing linear transformations. In
particular, we proved that for any number \( a \), and for any \( b > 0 \), the vNM utility
function \( v(x) = a + bu(x) \) represents the same preferences as \( u \).

6. (10 points). Suppose that Jeremy’s vNM utility function is \( u \), and suppose that
\( u(100) = 1 \), and \( u(0) = 0 \). Jeremy is offered a lottery:
\[
L = \begin{cases} 
100 & \text{w.p. } 0.6 \\
0 & \text{w.p. } 0.4 
\end{cases}
\]
Jeremy’s certainty equivalent to the lottery is 55.

a. Based on the above, Jeremy is risk-averse. True/false, prove your answer.

A risk averse person prefers the mean of the lottery to the lottery itself, and therefore
would pay less than 60 (the mean of \( L \)) for this lottery. Since 55 < 60, we know he is risk-
averse. The above also means that Jeremy is willing to pay a risk premium of 5 in order
to avoid the lottery and get its mean.

b. Find Jeremy’s utility from sure payoff of 55, i.e. find \( u(55) \).

From the definition of certainty equivalent,
\[
u(CE) = E[u(x)]
\]
\[
u(55) = 0.4u(0) + 0.6u(100) = 0.6
\]
7. (10 points). For the vNM utility function \( u(x) = -e^{-Ax} \), prove that the Arrow-Pratt coefficient of Absolute Risk Aversion is the constant \( A \).

\[
\begin{align*}
u'(x) &= Ae^{-Ax} \\
u''(x) &= -A^2e^{-Ax} \\
ARA(x) &= -\frac{A^2e^{-Ax}}{Ae^{-Ax}} = A
\end{align*}
\]

8. (10 points). Suppose that the vNM utility function \( u \) is quadratic, i.e. \( u(x) = x - bx^2, \ b > 0 \). Prove that there exists a mean-variance utility function \( v(\mu, \sigma) \), which describes the same preferences as \( u \).

\[
\begin{align*}
E[u(x)] &= E[x - bx^2] = E(x) - bE(x^2) \\
&= \mu - b(\sigma^2 + \mu^2) = v(\mu, \sigma)
\end{align*}
\]
9. (10 points). Consider the Prospect Theory value function in the next graph:

The above value function is (circle the correct answer):

i. Risk-seeking for gains and risk-averse of losses,
ii. Risk-seeking for losses and risk-averse for gains,
iii. Risk-seeking for gains and losses,
iv. Risk-averse for gains and losses.
10. (10 points). Consider a lottery $L$ in which a die is tossed and you receive the number on the die if it is even and pay the number on the die if it is odd. The outcomes are: -5, -3, -1, 2, 4, 6. Suppose that the prospect theory value function is $v(x)$, and probability weighting function is $w(p)$.

a. Write the original Prospect Theory utility from this lottery, $PT(L)$.

$$PT(L) = w \left( \frac{1}{6} \right) v(-5) + w \left( \frac{1}{6} \right) v(-3) + w \left( \frac{1}{6} \right) v(-1) + w \left( \frac{1}{6} \right) v(2) + w \left( \frac{1}{6} \right) v(4) + w \left( \frac{1}{6} \right) v(6)$$

b. Write the Cumulative Prospect Theory utility from this lottery, $CPT(L)$.

The cumulative probabilities are:

$$F_1 = \frac{1}{6}, F_2 = \frac{2}{6}, F_3 = \frac{3}{6}, F_4 = \frac{4}{6}, F_5 = \frac{5}{6}, F_6 = 1$$

Thus,

$$CPT(L) = \left[ w \left( \frac{1}{6} \right) - w(0) \right] v(-5) + \left[ w \left( \frac{2}{6} \right) - w \left( \frac{1}{6} \right) \right] v(-3) + \left[ w \left( \frac{3}{6} \right) - w \left( \frac{2}{6} \right) \right] v(-1) + \left[ w \left( \frac{4}{6} \right) - w \left( \frac{3}{6} \right) \right] v(2) + \left[ w \left( \frac{5}{6} \right) - w \left( \frac{4}{6} \right) \right] v(4) + \left[ w(1) - w \left( \frac{5}{6} \right) \right] v(6)$$