Neoclassical Growth Model

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1 Introduction

In these notes we describe one of the workhorses of modern macroeconomics - the Neoclassical Growth Model (NGM). This model is sometimes referred to as the Ramsey Model of Optimal Growth. The Solow model gave us some basic intuition about what factors are important for growth, but the Solow model lacks micro-foundations, in that consumers are assumed to use a rule of thumb for dividing income into consumption and saving, and everybody works full time. Thus, the Solow model does not have a role for consumers’ choices. In particular, we want to see how consumers respond to changes in economic environment, government policies, etc. We want to see how consumers change their saving behavior and their labor supply decisions as a response to changes in economic conditions. The neoclassical growth model will allow us to do exactly that.

Modern macroeconomic research on business cycles uses the neoclassical growth model, augmented with stochastic shocks to productivity - Stochastic NGM. In order to study the effects of monetary policy, macroeconomists introduce money into the NGM, and then the prices of all goods are nominal - in units of money. In order to allow monetary policy to have an effect on real variables in the economy, macroeconomists introduce price rigidities or wage rigidities in the model (i.e. relaxing the assumption that prices and wages always adjust immediately to equilibrium). Such models, with nominal rigidities, are called New Keynesian models. All these extensions of the NGM studied in these notes, are simulated and are used as a laboratory - a computational experiment, which gives quantitative results to well defined questions. For example, what caused the great depression? What would have happened if the Federal Reserve had reacted to stock price misalignments prior to crashes and major financial crises such as the Great Recession? The idea of the general methodology is described in Kydland, Finn E., and Edward C. Prescott 1996, "The Computational Experiment: An Econometric Tool", Journal of Economic Perspectives, 10(1): 69-85.

2 Deterministic NGM, No Government

1. Representative Household. There is a single household that lives forever, and enjoys consumption and leisure. The period utility function is $u(c_t, l_t)$, and the lifetime utility is $U \left( \{c_t, l_t\}_{t=0}^\infty \right) = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$. We assume that $u$ is strictly increasing and strictly concave.
The calculus conditions for these assumptions (provided that $u$ is twice-differentiable), are\(\text{\[a\]}\):
\[
(a) : u_1 (c, l) > 0, \ u_2 (c, l) > 0
\]
\(\text{\[b\]}\):
\[
(b) : u_{11} (c, l) < 0, \ u_{11} (c, l) u_{22} (c, l) - u_{12} (c, l)^2 > 0
\]

Here $\beta \in (0, 1)$ is a discount factor, which the household uses to discount future consumption\(\text{\[\#\]}\) . Households are endowed with one unit of time, which they can allocate between work, $h_t$, and leisure $l_t$. The household owns the capital stock $k_t$ and the firm, to be described below. In this version, the household makes the saving or investment decisions. Let $x_t$ be the amount of saving or investment at time $t$. Thus, the household’s income comes from work, renting the capital stock, and the dividends from the firm, $\pi_t$. The household’s problem is therefore:

\[
\max_{\{c_t, l_t, h_t, x_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u (c_t, l_t) \tag{1}
\]

\[
s.t.
\]

[Time constraint] : $h_t + l_t = 1 \ \forall t$

[Budget constraint] : $c_t + x_t = w_t h_t + r_t k_t + \pi_t \ \forall t$

[Law of motion of capital] : $k_{t+1} = (1 - \delta) k_t + x_t \ \forall t$

[Initial capital] : $k_0 > 0$ is given

[Non-negativity] : $c_t, l_t, h_t, x_t, k_{t+1} \geq 0 \ \forall t$

This is a generic constrained optimization problem. The function to be maximized, $\sum_{t=0}^{\infty} \beta^t u (c_t, l_t)$ is called the objective function, the variables under the word "max" are called the choice variables, the abbreviation s.t. means subject to or such that, and it is followed by the constraints. The left hand side of the budget constraint is the spending on consumption and investment, and the right hand side is the income from labor, capital and dividends. Consumption is the numéraire, and its price is normalized to 1. The wages $w_t$ are therefore expressed in real terms, i.e. units of consumption per unit of time\(\text{\[\#\]}\) , and $r_t$ is the rental rate of capital\(\text{\[\#\]}\).

2. **Representative Firm.** The total output is produced with the neoclassical production function

\[
Y_t = F \left( K_t, L_t \right)
\]

This means that $F$ satisfies all the properties of the neoclassical production function that

---

1\(\text{\[\#\]}\) Condition (b) means that the hessian matrix $H = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ is negative definite. A matrix is negative definite if and only if its leading principle minors alternate signs.

2\(\text{\[\#\]}\) You can think of $\beta = 1/(1 + \rho)$, where $\rho$ is some discount rate, which works like interest rate in calculating present values. Thus, the household wants to maximize present value of the stream of utilities.

3\(\text{\[\#\]}\) Another option would be to introduce money in this model. Then the price of consumption could be expressed in terms of money.

4\(\text{\[\#\]}\) If $r_t = 0.1$ for example, then the household is paid 0.1 units of consumption for each unit of capital it rents to the firm over a time period.
we discussed earlier. The firm’s problem is, for all \( t \)

\[
\max_{K_t, L_t} \pi_t = Y_t - r_t K_t - w_t L_t \\
\text{s.t.} \\
Y_t = F(K_t, L_t)
\]

The profit or the dividend is given to the household, but since the production function is CRS, we must have \( \pi_t = 0 \ \forall t \) in equilibrium\(^5\).

3. Competitive Equilibrium. A competitive equilibrium consists of prices \( \{w_t, r_t\}_{t=0}^{\infty} \) and allocations to the household \( \{c_t, l_t, h_t, x_t, k_{t+1}\}_{t=0}^{\infty} \) and to the firm \( \{K_t, L_t, Y_t\}_{t=0}^{\infty} \), such that:

(a) Given the prices \( \{w_t, r_t\}_{t=0}^{\infty} \) the allocation to the representative household \( \{c_t, l_t, h_t, x_t, k_{t+1}\}_{t=0}^{\infty} \) solves the household’s problem (1),

(b) Given the prices \( \{w_t, r_t\}_{t=0}^{\infty} \) the allocation to the representative firm \( \{K_t, L_t, Y_t\}_{t=0}^{\infty} \) solves the firm’s problem (2),

(c) Markets are cleared

\[
\text{[Goods market]} : Y_t = c_t + x_t \\
\text{[Labor market]} : h_t = L_t \\
\text{[Capital market]} : k_t = K_t
\]

In other words, supply = demand in all the markets.

3 \hspace{1em} Solving the model

The main difficulty in solving this model is with the household’s problem. Note that solving this problem entails finding infinitely many unknowns, \( \{c_t, l_t, h_t, x_t, k_{t+1}\}_{t=0}^{\infty} \), that maximize the objective function subject to infinitely many constraints. The firm’s problem is a simple static problem. The reason for that is the assumption that the household owns all the factors of production (labor and capital), and makes the investment decisions. It is a nice exercise to describe the environment and define competitive equilibrium in the case where the firm owns the capital and makes the investment decisions. This alternative setting turns out to be exactly equivalent to the original one, in the sense that equilibrium prices and equilibrium quantities are exactly the same.

3.1 \hspace{1em} Solving the household’s problem

We start with the household problem. First, since \( \pi_t = 0 \ \forall t \), we can omit this variable from the household’s problem. Next, substitute the time constraint into the objective function and substitute the law of motion of capital into the budget constraint. The simplified problem becomes

\[
\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\
\text{s.t.} \\
c_t + k_{t+1} = w_t h_t + r_t k_t + (1 - \delta) k_t \ \forall t
\]

\(^5\)This result is a special case of Euler’s theorem.
For simplicity of notation, we stop writing that $k_0 > 0$ is given and that the consumption, labor, capital must all be non-negative. But we bear in mind that these conditions are needed for solving the model numerically. The Lagrange function corresponding to the household’s problem is

$$L (\{c_t, h_t, k_{t+1}, \lambda_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) - \sum_{t=0}^{\infty} \lambda_t [c_t + k_{t+1} - w_t h_t - r_t k_t - (1 - \delta) k_t]$$

where $\{\lambda_t\}_{t=0}^{\infty}$ is the sequence of Lagrange multipliers. The first order conditions with respect to $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$

$$[c_t] : \beta^t u_1(c_t, 1 - h_t) - \lambda_t = 0 \ \forall t \quad (3)$$
$$[h_t] : -\beta^t u_2(c_t, 1 - h_t) + \lambda_t w_t = 0 \ \forall t \quad (4)$$
$$[k_{t+1}] : -\lambda_t + \lambda_{t+1} (r_{t+1} + 1 - \delta) = 0 \ \forall t \quad (5)$$

Thus, we have infinitely many first order conditions. Dividing (4) by (3) gives

$$\frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = w_t \ \forall t \quad (6)$$

This condition is the familiar condition for optimal leisure, which says that the marginal rate of substitution between leisure and consumption equals the price ratio. Recall that the price of consumption is normalized to 1.

Using (3) for period $t$ and $t + 1$ and (5) gives

$$\frac{\beta^t u_1(c_t, 1 - h_t)}{\beta^{t+1} u_1(c_{t+1}, 1 - h_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}} = r_{t+1} + 1 - \delta$$

$$\frac{u_1(c_t, 1 - h_t)}{u_1(c_{t+1}, 1 - h_{t+1})} = \beta (r_{t+1} + 1 - \delta) \ \forall t \quad (7)$$

The last equation is called Euler equation. The Euler equation contains an important intuition about the optimal saving (or investment) condition. The Euler equation can be written as

$$u_1(c_t, 1 - h_t) = \beta u_1(c_{t+1}, 1 - h_{t+1}) (r_{t+1} + 1 - \delta)$$

The left hand side is the "pain" from giving up one unit of consumption as the household invests one extra unit in future capital, i.e. increasing $k_{t+1}$. The right hand side is the "gain" from that investment. In particular, the future return on capital is $r_{t+1} + 1 - \delta$, which is the rate of return $r_{t+1}$ plus the non-depreciated unit of capital. Multiplying this return by $u_1$ translates the return into units of utility. Finally, since the return on investment is collected in the next period, the right hand side is multiplied by $\beta$ - the discount factor.

### 3.2 Solving the firm’s problem

The firm’s problem is fairly simple static problem. Substituting the technology constraint into the objective, gives the unconstrained problem

$$\max_{K_t, L_t} \pi_t = F(K_t, L_t) - r_t K_t - w_t L_t$$
The first order conditions are

\[
F_1(K_t, L_t) = r_t \\
F_2(K_t, L_t) = w_t
\]

These are the familiar conditions that state that competitive firms pay each input its marginal product.

### 3.3 Solving for Equilibrium

We need to combine the market clearing conditions with the above conditions for utility and profit maximization. Thus, the factor prices are

\[
F_1(k_t, h_t) = r_t \\
F_2(k_t, h_t) = w_t
\]

Next, the equilibrium in the final goods market can be written as

\[
c_t + x_t = F(k_t, h_t) \\
c_t + k_{t+1} = F(k_t, h_t) + (1 - \delta) k_t \quad \forall t
\]

To summarize, the equilibrium sequences of consumption, labor, and capital, must satisfy \( \forall t = 0, 1, 2, ... \)

\[
\frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = F_2(k_t, h_t) \\
\frac{u_1(c_{t+1}, 1 - h_{t+1})}{u_1(c_t, 1 - h_t)} = \beta \left[ F_1(k_{t+1}, h_{t+1}) + 1 - \delta \right] \\
c_t + k_{t+1} = F(k_t, h_t) + (1 - \delta) k_t
\]

In words, these are the condition for optimal time allocation \([9]\), the Euler equation \([10]\) and feasibility constraint \([11]\). The unknowns are \( \{c_t, h_t, k_{t+1}\}_{t=0}^{\infty} \). Observe that saving and investment in this economy is

\[
s_t = x_t = k_{t+1} - (1 - \delta) k_t
\]

and saving and investment rate is

\[
\frac{s_t}{F(k_t, h_t)} = \frac{x_t}{F(k_t, h_t)} = \frac{k_{t+1} - (1 - \delta) k_t}{F(k_t, h_t)}
\]

Notice that unlike the Solow model, the saving (and investment) rate in NGM may change over time, depending on productivity of the production function and the current level of capital.

Since it is not possible to solve for infinitely many unknowns, for numerical solutions we approximate the solution by solving a truncated problem for \( t = 0, 1, ..., T \), instead of \( t = 0, 1, 2, ... \). In other words, we truncate the time horizon to some large number, say \( T = 400 \), instead of infinite
horizon. The above system of first order and feasibility constraints \([9](T+1)\) becomes a system of
\[3 \times (T + 1)\] equations. Let’s now count the unknowns:
\[c_0, c_1, \ldots, c_T, c_{T+1} \text{ (} T + 2 \text{ unknowns)}
\]
\[h_0, h_1, \ldots, h_T, h_{T+1} \text{ (} T + 2 \text{ unknowns)}
\]
\[k_0, k_1, \ldots, k_T, k_{T+1} \text{ (} T + 2 \text{ unknowns)}
\]
This makes the number of unknowns \(3 \times (T + 2)\). Does this mean that we are doomed, since
we have 3 extra unknowns without equations? Well, not exactly. Recall that \(k_0\) is given, so that
makes the number of unknowns to be \(3 \times (T + 1) + 2\). We still need two more equations. If the
model converges to a steady state \((c_{ss}, h_{ss}, k_{ss})\) and if \(T\) is big enough, then we can add two more
equations
\[c_{T+1} = c_{ss}
\]
\[h_{T+1} = h_{ss}
\]
Alternatively, if we did not compute the steady state, the two extra equations can be
\[c_{T+1} = c_T
\]
\[h_{T+1} = h_T
\]
It turns out that without technological progress, and under some restrictions on the utility function,
the NGM converges to a steady state, starting from any initial capital \(k_0 > 0\). This is similar to
proposition 1 in the notes on Solow model.

4 Quantitative Analysis

Let us solve the NGM for a special case of Cobb-Douglas utility function and production function,
i.e.
\[u(c_t, 1 - h_t) = \alpha \ln (c_t) + (1 - \alpha) \ln (1 - h_t)
\]
\[F(k_t, h_t) = A_t k_t^{\theta} h_t^{1-\theta}, \quad 0 < \theta < 1
\]
The first order and feasibility conditions become \(\forall t = 0, 1, 2, \ldots\)
\[
\left(\frac{1 - \alpha}{\alpha}\right) \frac{c_t}{1 - h_t} = (1 - \theta) A_t k_t^{\theta} h_t^{1-\theta} \quad (12)
\]
\[
\frac{c_{t+1}}{c_t} = \beta \left[ \theta A_{t+1} k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + 1 - \delta \right] \quad (13)
\]
\[c_t + k_{t+1} = A_t k_t^{\theta} h_t^{1-\theta} + (1 - \delta) k_t \quad (14)
\]
4.1 Steady state
If $A_t = A \forall t$, then the above system converges to a steady state. This statement is not proved here. Using $(c_t, h_t, k_t) = (c_{ss}, h_{ss}, k_{ss}) \forall t$, gives a system of 3 equations with 3 unknowns

$\left( \frac{1 - \alpha}{\alpha} \right) c_{ss} = (1 - \theta) A^{1 - \theta} k_{ss}^{1 - \theta}$ \hspace{1cm} (15)

$1 = \beta [\theta A k_{ss}^{\theta - 1} h_{ss}^{1 - \theta} + 1 - \delta]$ \hspace{1cm} (16)

$c_{ss} = A k_{ss}^{\theta - 1} h_{ss}^{1 - \theta} - \delta k_{ss}$ \hspace{1cm} (17)

This system can be solved analytically, but usually a numerical solution is applied. See the appendix for the analytical solution. The steady state output, wages and rate of return to capital are given by

$y_{ss} = A^{1 - \theta} k_{ss}^{1 - \theta}$ \hspace{1cm} (18)

$w_{ss} = (1 - \theta) A k_{ss}^{\theta - 1} h_{ss}^{1 - \theta} = (1 - \theta) \frac{y_{ss}}{h_{ss}}$ \hspace{1cm} (19)

$r_{ss} = \theta A k_{ss}^{\theta - 1} h_{ss}^{1 - \theta} = \theta \frac{y_{ss}}{k_{ss}}$ \hspace{1cm} (20)

Also notice that saving and investment at steady state is

$s_{ss} = x_{ss} = \delta k_{ss}$

and saving or investment rate is

$s = \frac{s_{ss}}{y_{ss}} = \frac{x_{ss}}{y_{ss}} = \frac{\delta k_{ss}}{y_{ss}}$

4.2 Optimal saving rate at steady state
Recall that in the Solow model the optimal saving rate is the one which maximizes the steady state level of consumption per worker, and is called the golden rule saving rate. In the Solow model with Cobb-Douglas production function we found that $s_{GR} = \theta$. In the Neoclassical Growth Model the equilibrium saving rate is optimal by construction of the model - the household chooses consumption optimally in order to maximize the lifetime utility\footnote{Recall that in the Solow model consumers use a rule of thumb when choosing how much to save and how much to consume.}. Let us compare the steady state saving rate in the NGM with the golden rule saving rate in the Solow model. The steady state saving (and investment) rate from the feasibility constraint in equation (17) is

$s = \frac{\delta k_{ss}}{y_{ss}} \Rightarrow \frac{y_{ss}}{k_{ss}} = \frac{\delta}{s}$
Using this and $\beta = 1/(1 + \rho)$ in equation (16) gives

\[ 1 = \beta \left[ \frac{y_{ss}}{k_{ss}} + 1 - \delta \right] \]
\[ 1 + \rho = \frac{\delta}{s} \]
\[ \rho + \delta = \frac{\delta}{s} \]
\[ s = \left( \frac{\delta}{\rho + \delta} \right) \theta \]

First notice that if $\rho = 0$, then the optimal saving rate in the NGM is the same as in the Solow model: $s = s_{GR} = \theta$. Recall that $\rho$ is a discount rate that the household uses to discount future utility. Higher values of $\rho$ mean that the household is more impatient or cares about the future less (lives the day). Therefore, the optimal saving (and investment) rate is lower since the household doesn’t want to invest that much in future capital. Consider a case of an economy with capital share $\theta = 0.35$, depreciation rate of $\delta = 0.05$ and discount rate of $\rho = 0.04$. Based on the Solow growth model, the optimal saving rate in this economy is $s_{GR} = 35\%$. However, based on the NGM, the optimal saving rate is much lower.

\[ s = \left( \frac{0.05}{0.04 + 0.05} \right) 0.35 = 19.44\% \]

In fact, the maximal possible steady state saving rate in NGM is $\theta$. Suppose that in China, people care about the future more than in the U.S., so the Chinese time discount rate is $\rho = 0.02$. The saving rate in China will be

\[ s_{China} = \left( \frac{0.05}{0.02 + 0.05} \right) 0.35 = 25\% \]

We can also see that the optimal saving rate is increasing in depreciation rate $\delta$. This makes intuitive sense. If depreciation rate is $\delta = 0$, then capital never depreciates and we don’t need to save or invest to maintain the steady state capital. With higher depreciation rate, maintaining the steady state capital requires higher investment rate. If we change the depreciation rate to $\delta = 0.1$ in the above example, the steady state saving rate becomes:

\[ s = \left( \frac{0.1}{0.02 + 0.1} \right) 0.35 = 0.29\% \]

### 4.3 Balanced growth path

Recall that we proved that in the Solow model, when productivity grows at constant rate, there exists a unique Balanced Growth Path, such that all endogenous variables grow at constant rate. The BGP is consistent with certain facts about modern growth experience in developed countries. Similarly, in the Neoclassical Growth Model, there exists a unique BGP, when productivity grows at constant rate. The next proposition characterizes the Balanced Growth Path in the Neoclassical Growth Model.
Proposition 1 Consider the Neoclassical Growth Model with Cobb-Douglas utility and production. Suppose that TFP grows at constant rate $\gamma_A$, i.e. $A_t = A_0 (1 + \gamma_A)^t$. Then on a BGP, consumption, output, capital and wage, grow at constant rate $\gamma$, given below:

\[
\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{w_{t+1}}{w_t} = 1 + \gamma = (1 + \gamma_A)^{\frac{1}{1-\theta}}
\]

Also, the labor and rate of return on capital along the BGP are fixed: $r_t = r$, $h_t = h$.

Proof. The proof uses equilibrium equations (12)-(14). Let the BGP growth rates of $c_t$, $y_t$ and $k_t$ be $\gamma_c$, $\gamma_y$ and $\gamma_k$ respectively. Using the Euler equation, (13), we can see that output and capital grow at the same rate:

\[
\frac{c_{t+1}}{c_t} = \beta \left[ \theta A_{t+1} k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + 1 - \delta \right]
\]

\[
1 + \gamma_c = \beta \left[ \theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right]
\]

Since the left hand side is constant, we have that the right hand side must be constant for all $t$, and $y_{t+1}/k_{t+1} = \eta_{yk}$ (constant) implies that $\gamma_y = \gamma_k = \gamma$. This also implies that $r_t = \theta \eta_{yk} = r$ (constant) on a BGP.

Using the feasibility constraint, equation (14), and dividing by $k_t$ and rearranging, gives:

\[
c_t + k_{t+1} = A_t k_t^\theta h_t h_{t+1}^{1-\theta} + (1 - \delta) k_t
\]

\[
\frac{c_t}{k_t} + 1 + \gamma = \frac{y_t}{k_t} + 1 - \delta
\]

\[
\frac{c_t}{k_t} + \gamma = \eta_{yk} - \delta
\]

Thus, $c_t/k_t = \eta_{ck}$ (constant) on a BGP, implying that $\gamma_c = \gamma_y = \gamma_k = \gamma$.

Dividing equation (12) by $c_t$ and multiplying by $h_t$ gives:

\[
\left( \frac{1 - \alpha}{\alpha} \right) \frac{c_t}{1 - h_t} = (1 - \theta) A_t k_t^\theta h_t^{1-\theta}
\]

\[
\left( \frac{1 - \alpha}{\alpha} \right) \frac{h_t}{1 - h_t} = (1 - \theta) \frac{A_t k_t^\theta h_t^{1-\theta}}{c_t}
\]

\[
\left( \frac{1 - \alpha}{\alpha} \right) \frac{h_t}{1 - h_t} = (1 - \theta) \frac{y_t}{c_t} = (1 - \theta) \eta_{yc}
\]

where $\eta_{yc} = y_t/c_t$ (constant). The last step follows since we already proved that consumption and output grow at the same rate, so their ratio must be constant. The last equation implies that $h_t = h$ (constant) on a BGP.

The wage $w_t$ is given by

\[
w_t = (1 - \theta) A_t k_t^\theta h_t^{1-\theta} = \left( \frac{1 - \theta}{h} \right) A_t k_t^\theta h_t^{1-\theta} = \left( \frac{1 - \theta}{h} \right) y_t
\]
The second step uses the proven constancy of \( h_t \) on a BGP. Thus, \( w_t \) is proportional to output and must grow at the same rate as \( y_t \).

Finally, by constancy of rate of return on capital on BGP, we have:

\[
\frac{r_{t+1}}{r_t} = \frac{\theta A_{t+1} k_{t+1}^{\theta-1} h_{t+1}^{1-\theta}}{\theta A_t k_t^{\theta-1} h_t^{1-\theta}} = \frac{A_{t+1} k_{t+1}^{\theta-1}}{A_t k_t^{\theta-1}} = (1 + \gamma_A) (1 + \gamma)^{\theta-1} = 1
\]

\[
\Rightarrow 1 + \gamma = (1 + \gamma_A)^{\frac{1}{\theta-1}}
\]

which is the same as in the Solow model. ■

The above proposition allows us to write the BGP equilibrium equations in terms of constant ratios of variables which grow at the same rate:

\[
\text{BGP equations:}
\]

\[
\left( \frac{1 - \alpha}{\alpha} \right) \frac{1}{h} = (1 - \theta) \eta_{yc} \tag{21}
\]

\[
1 + \gamma = \beta \left[ \theta \eta_{yk} + 1 - \delta \right] \tag{22}
\]

\[
\eta_{ck} + \gamma = \eta_{yk} - \delta \tag{23}
\]

4.4 Calibration

We typically calibrate the model to match the growth trends of developed economies. Such economies typically exhibit balanced growth, with business cycles. We need to calibrate the parameters \( \alpha, \theta, \gamma \) (or \( \gamma_A \)), \( \delta \) and \( \beta \). The parameter \( \gamma \) can be calibrated directly from the data, as the average growth rate of real consumption per capita or real GDP per capita. As we discussed before, the capital share in the data is approximately 35%, so this pins down \( \theta = 0.35 \). The remaining 3 parameters can be calibrated from the BGP equations \( (21)-(22) \), if indeed the ratios \( \frac{y_t}{k_t}, \frac{c_t}{k_t}, \frac{y_t}{c_t} \) are more or less constant in the data. Using the data averages for \( \frac{y_t}{k_t}, \frac{c_t}{k_t} \) and \( \frac{y_t}{c_t} \), we obtain values for \( \eta_{yk}, \eta_{ck} \) and \( \eta_{yc} \). Equation \( (23) \) pins down \( \delta \) (since we already found \( \gamma \)). Alternatively, \( \delta \) can be measured directly, as depreciation rate of fixed assets (physical capital). Equation \( (22) \) allows us to calibrate \( \beta \). Finally, equation \( (21) \) can be used to calibrate \( \alpha \). Suppose that there are 100 hours of discretionary time per week \((1 \cdot (24 - 8 \text{ hours of sleep} - 2 \text{ personal care}) = 98 \approx 100)\). A workweek of 40 hours means that \( h = 0.4 \). Thus,

\[
\frac{h}{1 - h} = \frac{0.4}{0.6} = \frac{2}{3}
\]

Then, using equation \( (21) \), gives

\[
\left( \frac{1 - \alpha}{\alpha} \right) \frac{2}{3} = (1 - 0.35) \eta_{yc}
\]

with the only unknown \( \alpha \).

Example 2 From the U.S. data, we get \( \eta_{yk} = 0.3, \eta_{yc} = 1.5, \gamma = 3\% \). This implies that

\[
\eta_{ck} = \frac{c_t}{k_t} = \frac{c_t y_t}{y_t k_t} = \frac{1}{1.5} \cdot 0.3 = 0.2
\]
From (23) we get

\[ \eta_{ck} + \gamma = \eta_{yk} - \delta \]
\[ \delta = \eta_{yk} - \eta_{ck} - \gamma \]
\[ \delta = \sqrt{\frac{\eta_{yk}}{0.3}} - \frac{\eta_{ck}}{0.2} - \frac{\gamma}{0.03} = 0.07 \]

Alternatively, the annual depreciation rate in the data is about \( \delta = 6\% \). This is preferred estimate, since it is coming directly from the data, while the depreciation rate estimated based on (23) is very sensitive to other estimates \((\eta_{yk}, \eta_{ck})\). From (22) we get

\[ 1 + \gamma = \beta [\theta \eta_{yk} + 1 - \delta] \]
\[ 1 + 0.03 = \beta [0.35 \cdot 0.3 + 1 - 0.06] \]
\[ \beta = 0.98565 \]

Finally, using \( \eta_{yc} = 1.5 \) and \( h = 0.4 \) in equation (21), gives

\[ \left( \frac{1 - \alpha}{\alpha} \right) \frac{h}{1 - h} = (1 - \theta) \eta_{yc} \]
\[ \alpha \approx 0.4 \]

### 4.5 Optimal saving rate on a BGP

We would like to solve for the saving rate on the BGP in terms of the model’s parameters. The law of motion of capital is:

\[ k_{t+1} = (1 - \delta) k_t + x_t \]
\[ \frac{k_t}{y_t} (1 + \gamma) = (1 - \delta) \frac{k_t}{y_t} + \frac{x_t}{y_t} \]
\[ s = \frac{\gamma + \delta}{\eta_{yk}} \]

where \( s = x_t/y_t \) is the saving and investment rate. Notice that on a BGP, the saving rate is constant. We want to express the saving rate in terms of the deep parameters of the model. Solving for \( \eta_{yk} \) from (22) gives:

\[ 1 + \gamma = \beta [\theta \eta_{yk} + 1 - \delta] \]
\[ (1 + \gamma) (1 + \rho) = \theta \eta_{yk} + 1 - \delta \]
\[ 1 + \gamma + \rho + \gamma \rho = \theta \eta_{yk} + 1 - \delta \]
\[ \eta_{yk} = \frac{\delta + \gamma + \rho + \gamma \rho}{\theta} \]

Plugging into the expression for saving rate, gives:

\[ s = \frac{\gamma + \delta}{\eta_{yk}} = \frac{\gamma + \delta}{\delta + \gamma + \rho + \gamma \rho} \theta \]
First observe that when \( \gamma = 0 \), we have the steady state saving rate as a special case:

\[
s = \frac{0 + \delta}{\delta + 0 + \rho + 0} \theta = \frac{\delta}{\delta + \rho}
\]

In general, we can see that the BGP saving rate increases in \( \gamma \):

\[
\frac{\partial s}{\partial \gamma} = \frac{\delta + \gamma + \rho + \gamma \rho - (1 + \rho)(\delta + \gamma)}{(\delta + \gamma + \rho + \gamma \rho)^2} \theta
\]

\[
= \frac{\delta + \gamma + \rho + \gamma \rho - \delta - \gamma - \rho \delta - \gamma \rho}{(\delta + \gamma + \rho + \gamma \rho)^2} \theta
\]

\[
= \frac{\rho (1 - \delta)}{(\delta + \gamma + \rho + \gamma \rho)^2} \theta > 0
\]

Intuitively, faster productivity growth \( \gamma_A \), and therefore faster \( \gamma \), means that capital becomes increasingly more productive and the return to investment (=saving) is higher. Therefore, the optimal investment rate (=saving rate) is higher. This offers another explanation for why China has higher saving and investment rate than the U.S. To summarize, in the Neoclassical Growth Model, the saving rate is determined endogenously, and depends on preferences and technology, while in the Solow model the saving rate is exogenous.

**Example 3 The high Chinese saving rate puzzle.** Suppose that U.S. and China have the following parameters:

<table>
<thead>
<tr>
<th></th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( \rho )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U.S.</strong></td>
<td>0.01</td>
<td>0.06</td>
<td>0.04</td>
<td>0.35</td>
</tr>
<tr>
<td><strong>China</strong></td>
<td>0.1</td>
<td>0.06</td>
<td>0.02</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Then, the saving rate in the two countries is:

\[
s_{U.S.} = \frac{\gamma + \delta}{\delta + \gamma + \rho + \gamma \rho} \theta = \frac{0.02 + 0.06}{0.06 + 0.01 + 0.04 + 0.01 \cdot 0.04 \cdot 0.35} = 25.36\%
\]

\[
s_{China} = \frac{\gamma + \delta}{\delta + \gamma + \rho + \gamma \rho} \theta = \frac{0.1 + 0.06}{0.06 + 0.1 + 0.02 + 0.1 \cdot 0.02 \cdot 0.35} = 30.77\%
\]

From the above example we see that even big differences in growth rates, such as 2% v.s. 10%, account for only small differences in saving or investment rate. In 2017, the saving rates of China and U.S. were: 45.8% and 17.5% respectively. Notice that the NGM is unable to account for such high saving rate in China, if we assume that \( \theta = 0.35 \) in both countries. Even if the discount factor in China is \( \rho = 0 \), i.e. the future is as important as the present, the maximal saving rate on a BGP in this model is \( \theta = 35\% \). Therefore we conclude that either (i) the capital share \( \theta \) in China is much higher, or (ii) China does not experience balanced growth (so our equation for BGP saving rate does not apply to China), or (iii) some components which are not included in our model (e.g. taxes) can account for the high saving rate.

### 4.6 Using the model

This model is not particularly interesting. The only question that we can ask is what is the effect of TFP and its growth on endogenous variables. The effect of TFP level on steady state is discussed
in appendix. In order to analyze the effect of TFP on the time path of endogenous variables, we need to solve numerically (using Matlab) the system (12)-(14). The next section discusses a model with various taxes, and is much more interesting.

5 NGM with Taxes

We now introduce a government into the model in order to study the effects of various fiscal policies on equilibrium outcomes. In this chapter we treat government policies as exogenous, and do not attempt to model the government’s objective. The government imposes the following taxes:

1. \( \tau_{ct} \) on \( c_t \)
2. \( \tau_{xt} \) on \( x_t \)
3. \( \tau_{wt} \) on \( w_t h_t \) (i.e. on labor income)
4. \( \tau_{kt} \) on \( r_t k_t \) (i.e. on capital income)

The government uses the taxes to finance its expenditures \( g_t \) and lump-sum transfers \( \tau_t \). Thus, in each period, the government’s income is \( \tau_{ct} c_t + \tau_{xt} x_t + \tau_{wt} w_t h_t + \tau_{kt} r_t k_t \) and its outlays are \( g_t + \tau_t \). For simplicity, we will assume that the government always balances its budget:

\[
g_t + \tau_t = \tau_{ct} c_t + \tau_{xt} x_t + \tau_{wt} w_t h_t + \tau_{kt} r_t k_t \quad \forall t
\]

The household’s problem in (11) is now modified to include taxes

\[
\max_{\{c_t, h_t, x_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\
\text{s.t.} \\
(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) x_t = (1 - \tau_{wt}) w_t h_t + (1 - \tau_{kt}) r_t k_t + \tau_t \\
k_{t+1} = (1 - \delta) k_t + x_t
\]

Notice that \( \tau_{ct} \) and \( \tau_{xt} \) are sales taxes, so the prices of consumption and investment increase as a result of taxes. The income taxes \( \tau_{wt} \) and \( \tau_{kt} \) on the other hand, reduce the income available for spending. Plugging the law of motion of capital into the budget constraint, simplifies the household’s problem to

\[
\max_{\{c_t, h_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\
\text{s.t.} \\
(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) k_{t+1} = (1 - \tau_{wt}) w_t h_t + (1 - \tau_{kt}) r_t k_t + (1 + \tau_{xt}) (1 - \delta) k_t + \tau_t
\]

Notice that if this household receives an extra unit of capital at time \( t \), his income would increase by \( (1 - \tau_{kt}) r_t + (1 + \tau_{xt}) (1 - \delta) \). This magnitude is called the value of unit of capital, and we will denote it by \( p_{kt} \):

\[
p_{kt} \equiv (1 - \tau_{kt}) r_t + (1 + \tau_{xt}) (1 - \delta)
\]

The firm’s problem remains unchanged, and is not repeated here.
5.1 Tax Distorted Competitive Equilibrium (TDCE)

A TDCE, for given sequence of fiscal policies \(\{g_t, \tau_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty}\), consists of prices \(\{w_t, r_t\}_{t=0}^{\infty}\) and allocations to the household \(\{c_t, l_t, h_t, x_t, k_{t+1}\}_{t=0}^{\infty}\) and to the firm \(\{K_t, L_t, Y_t\}_{t=0}^{\infty}\) such that:

1. Given the prices \(\{w_t, r_t\}_{t=0}^{\infty}\) and fiscal policies \(\{g_t, \tau_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty}\), the allocation to the representative household \(\{c_t, l_t, h_t, x_t, k_{t+1}\}_{t=0}^{\infty}\) solves the household’s problem (24),

2. Given the prices \(\{w_t, r_t\}_{t=0}^{\infty}\) and fiscal policies \(\{g_t, \tau_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty}\), the allocation to the representative firm \(\{K_t, L_t, Y_t\}_{t=0}^{\infty}\) solves the firm’s problem (2).

3. Markets are cleared

\[
\begin{align*}
\text{[Goods market]} &: \quad Y_t = c_t + x_t + g_t \\
\text{[Labor market]} &: \quad h_t = L_t \\
\text{[Capital market]} &: \quad k_t = K_t
\end{align*}
\]

4. Government budget is balanced

\[g_t + \tau_t = \tau_{ct}c_t + \tau_{xt}x_t + \tau_{wt}w_lh_t + \tau_{kt}r_lk_t\]

**Remark.** It is easy to show that household’s budget constraint, together with the feasibility constraint, imply that the government budget is balanced. Rearranging the household’s budget constraint:

\[
(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) x_t = (1 - \tau_{wt}) w_lh_t + (1 - \tau_{kt}) r_lk_t + \tau_t
\]

\[
c_t + x_t = w_lh_t + r_lk_t - [\tau_{ct}c_t + \tau_{xt}x_t + \tau_{wt}w_lh_t + \tau_{kt}r_lk_t] + \tau_t
\]

\[
c_t + x_t = Y_t - [\tau_{ct}c_t + \tau_{xt}x_t + \tau_{wt}w_lh_t + \tau_{kt}r_lk_t] + \tau_t
\]

\[
c_t + x_t = c_t + x_t + g_t - [\tau_{ct}c_t + \tau_{xt}x_t + \tau_{wt}w_lh_t + \tau_{kt}r_lk_t] + \tau_t
\]

\[
g_t + \tau_t = \tau_{ct}c_t + \tau_{xt}x_t + \tau_{wt}w_lh_t + \tau_{kt}r_lk_t
\]

Thus, the lump-sum transfer \(\tau_t\) is always set in such a way that the government budget is balanced. For example, if \(g_t = 10\) and government’s revenue from all the taxes is 7, then \(\tau_t = -3\), i.e. the government must collect extra lump-sum taxes in order to finance its spending. Intuitively, this is a closed economy, so the government cannot borrow from abroad. We also do not have government bonds in this model, so the government cannot borrow from the consumer by selling bonds. Our main goal in this chapter is to study the effects of taxes on consumer’s choices, while debt and deficit issues are outside of our scope.

5.2 Solving for TDCE

As before, we start by deriving the optimality conditions for the household’s problem.

\[
\max \quad \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)
\]

s.t.

\[
(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) k_{t+1} = (1 - \tau_{wt}) w_lh_t + (1 - \tau_{kt}) r_lk_t + (1 + \tau_{xt})(1 - \delta) k_t + \tau_t
\]
The lagrange function is

\[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) - \sum_{t=0}^{\infty} \lambda_t \left[ (1 + \tau_{ct}) c_t + (1 + \tau_{xt}) k_{t+1} \right. \\
\left. - (1 - \tau_{wt}) w_t h_t - (1 - \tau_{kt}) r_t k_t - (1 + \tau_{xt}) (1 - \delta) k_t - \tau_t \right] \]

The first order necessary conditions with respect to \( \{c_t, h_t, k_{t+1}\}_{t=0}^{\infty} \) are

\[
\begin{align*}
[c_t] & : \beta^t u_1(c_t, 1 - h_t) - \lambda_t (1 + \tau_{ct}) = 0 \\
[h_t] & : -\beta^t u_2(c_t, 1 - h_t) + \lambda_t (1 - \tau_{wt}) w_t = 0 \\
[k_{t+1}] & : -\lambda_t (1 + \tau_{xt}) + \lambda_{t+1} [(1 - \tau_{kt+1}) r_{t+1} + (1 + \tau_{xt+1}) (1 - \delta)] = 0
\end{align*}
\]

Rearranging these conditions yields the intra-temporal and inter-temporal optimality conditions

\[
\begin{align*}
\frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} &= \frac{(1 - \tau_{wt})}{(1 + \tau_{ct})} w_t \tag{25} \\
u_1(c_t, 1 - h_t) \frac{(1 + \tau_{xt})}{(1 + \tau_{ct})} &= \beta u_1(c_{t+1}, 1 - h_{t+1}) \frac{(1 - \tau_{xt+1}) r_{t+1} + (1 + \tau_{xt+1}) (1 - \delta)}{(1 + \tau_{ct+1})} \tag{26}
\end{align*}
\]

These conditions are analogous to (6) and (7), but now the prices are distorted with taxes. Equation (25) governs the optimal time allocation. The left hand side is the marginal rate of substitution between leisure and consumption, while the right hand side is the relative price of leisure. Notice that the price of leisure is the after tax wage \( (1 - \tau_{wt}) w_t \) and the price of consumption is \( (1 + \tau_{ct}) \). Equation (26) is the familiar Euler equation. The left hand side is the "pain" from increasing investment by 1 unit. The household has to pay now \( (1 + \tau_{xt}) \) for one unit of investment and he gives up \( (1 + \tau_{xt}) / (1 + \tau_{ct}) \) units of consumption as a result. The "pain" is the loss of utility from that extra unit of investment - \( u_1(c_t, 1 - h_t) (1 + \tau_{ct}) / (1 + \tau_{ct}) \). On the right hand side we have the future gain from the extra investment. Notice that taxes distort the return on capital. The rate of return is \( (1 - \tau_{kt+1}) r_{t+1} \), but on the bright side, the non-depreciated unit of capital is worth more because the price of capital next period is \( (1 + \tau_{xt+1}) \).

It is convenient to write the Euler equation in a more compact way:

\[
\begin{align*}
u_1(c_t, 1 - h_t) &= \beta u_1(c_{t+1}, 1 - h_{t+1}) R_{t+1} \\
\text{where } R_{t+1} &= \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} \left[ \frac{(1 - \tau_{kt+1})}{(1 + \tau_{ct})} r_{t+1} + \frac{(1 + \tau_{xt+1})}{(1 + \tau_{xt})} (1 - \delta) \right] \tag{27}
\end{align*}
\]

The term \( R_{t+1} \) is the after-tax gross real interest rate between period \( t \) and \( t + 1 \). Recall the two-period consumption and saving model from undergraduate intermediate micro and macro courses:

\[
\max_{c_1,c_2} u(c_1) + \beta u(c_2) \\
\text{s.t. } c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r}
\]

The optimality condition in this model is

\[ u'(c_1) = \beta u'(c_2) (1 + r) \]
where $r$ is the real interest rate. The exact same condition holds in our multi-period model, with $R$ replacing $1 + r$. Recall that $1 + r$ is the relative price of current consumption ($c_1$) in terms of future consumption ($c_2$). Indeed, when we consume one more unit of $c_1$, we give up $1 + r$ units of $c_2$ (i.e., the opportunity cost of 1 unit of current consumption is $1 + r$ units of future consumption).

**Exercise 1** Show that $R_{t+1}$ from (27) is the relative price $c_t$ in terms of $c_{t+1}$. Hint: increase $c_t$ by 1 unit, and show that the household must give up $R_{t+1}$ units of $c_{t+1}$.

**Solution 1** Omitted.

We will investigate the effects of different tax policies on the equilibrium, using the same approach we used in Solow and NGM models to investigate exogenous changes in parameters. As before, if taxes are fixed and there is no technological progress, the model converges to a steady state. We can solve the model for $t = 0, 1, \ldots, T$, where $T$ is some large number of periods that are enough for the model to converge close enough to the steady state. After combining with the firm’s optimality $r_t = F_1 (k_t, h_t)$, $w_t = F_2 (k_t, h_t)$, the equilibrium path $\{c_t, h_t, k_{t+1}\}_{t=0}^\infty$ has to satisfy the following conditions for all $t$:

\[
\text{[Optimal labor]} : \quad \frac{u_2 (c_t, 1 - h_t)}{u_1 (c_t, 1 - h_t)} = \frac{(1 - \tau_{wt})}{(1 + \tau_{cd})} F_2 (k_t, h_t)
\]

\[
\text{[EE]} : \quad u_1 (c_t, 1 - h_t) = \beta u_1 (c_{t+1}, 1 - h_{t+1}) R_{t+1}
\]

where $R_{t+1} = \frac{(1 + \tau_{cd})}{(1 + \tau_{ct+1})} \left[ \frac{1 - \tau_{kt+1}}{1 + \tau_{xt}} F_1 (k_{t+1}, h_{t+1}) + \frac{1 + \tau_{xt+1}}{1 + \tau_{xt}} (1 - \delta) \right]

\[
\text{[Feas]} : \quad c_t + k_{t+1} + g_t = F (k_t, h_t) + (1 - \delta) k_t
\]

Suppose that fiscal policy is fixed over time: $\{g_t, \tau_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^\infty = (g, \tau, \tau_c, \tau_x, \tau_w, \tau_k) \forall t$ and suppose that all variables are at their steady state. Then the above system becomes

\[
\text{[Optimal labor]} : \quad \frac{u_2 (c, 1 - h)}{u_1 (c, 1 - h)} = \frac{(1 - \tau_w)}{(1 + \tau_c)} F_2 (k, h)
\]

\[
\text{[EE]} : \quad 1 = \beta \left[ \frac{1 - \tau_k}{1 + \tau_x} F_1 (k, h) + 1 - \delta \right]
\]

\[
\text{[Feas]} : \quad c + \delta k + g = F (k, h)
\]

6 Effects of taxes on equilibrium allocations and prices

6.1 Distorting taxes

**Definition 4** Distorting taxes prevent the competitive equilibrium allocation from solving the social planner’s problem

\[
\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u (c_t, 1 - h_t)
\]

s.t.

\[
c_t + k_{t+1} + g_t = F (k_t, h_t) + (1 - \delta) k_t
\]
The solution to the social planner’s problem is a sequence of endogenous variables \{c_t, h_t, k_t\}_{t=0}^{\infty}, which satisfies the following conditions:

\[
\begin{align*}
\frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} &= F_2(k_t, h_t) \\
\frac{u_1(c_t, 1-h_t)}{u_1(c_{t+1}, 1-h_{t+1})} &= \beta \frac{F_1(k_{t+1}, h_{t+1}) + 1 - \delta}{F_1(k_t, h_t)} \\
c_t + k_{t+1} + g_t &= F(k_t, h_t) + (1 - \delta) k_t
\end{align*}
\]

(31) (32) (33)

These conditions are exactly the same as equilibrium conditions in a model without taxes, (9)-(11). Thus, the competitive equilibrium in the model without government and without taxes is Pareto-Optimal - coincides with solution to social planner’s problem. Also notice that equilibrium with government and only lump-sum taxes is also Pareto-Optimal; set all the taxes to zero in TDCE conditions (28)-(30), and compare these conditions with the social planner’s (9)-(11). The lump-sum transfer \(\tau_d\) does not appear in the first order conditions, so if it was the only tax, the TDCE conditions would coincide with the first order conditions for the above social planner’s problem, and in this case, TDCE is Pareto Efficient. This is one of the major results in public finance - the most efficient way to collect given revenues for the government is through lump-sum taxes. The problem however is that from practical point of view, lump-sum taxes are difficult to implement.

6.2 Temporary and permanent effects of fiscal policies

By temporary effects we mean effects on the time path of equilibrium variables without changing the steady state, while permanent effect is such that the steady state (limiting behavior) of the model changes. Some tax policies will have only temporary effect, while others will alter the steady state. We can only perform limited analysis with pencil and paper, but fortunately our Matlab codes enable us to investigate the effects of a host of fiscal policies. Before we resort to Matlab however, lets try to develop some intuition about the possible effects of fiscal policies.

Once again the sufficient conditions for equilibrium path \{c_t, h_t, k_t\}_{t=0}^{\infty}, are:

[Optimal labor] : \frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = \frac{(1 - \tau_w)}{(1 + \tau_c)} F_2(k_t, h_t)

[EE] : u_1(c_t, 1-h_t) = \beta u_1(c_{t+1}, 1-h_{t+1}) R_{t+1}

[Feas] : c_t + k_{t+1} + g_t = F(k_t, h_t) + (1 - \delta) k_t

where \(R_{t+1} = \frac{(1 + \tau_c)}{(1 + \tau_c + 1)} \left[ \frac{1 - \tau_{k_t+1}}{1 + \tau_c + 1} \right] F_1(k_{t+1}, h_{t+1}) + \frac{1 + \tau_{x_t+1}}{1 + \tau_c + 1} (1 - \delta) \]

The willingness of households to substitute between consumption and leisure is affected by \(\tau_w\), \(\tau_c\) and wages. Higher taxes on labor (\(\tau_w\uparrow\)) reduce the after tax wage of the household, so the leisure becomes cheaper (substitution effect). This effect alone would make the household work less. This is not the only way the taxes on labor income affect the hours worked. Recall the lump-sum transfers have to adjust to balance the government budget. Since we did not say anything about changes in government spending, the entire increase in tax revenues will be given back to the household in the form of lump-sum transfers. This means a positive income effect, which on its
own leads to an increase in leisure (a normal good). On top of that, there is a general equilibrium effect since the equilibrium wage (marginal product of labor) will change. Hence without solving a parametrized version of the model numerically, we can never be sure of the effect of labor taxes on equilibrium labor $h_t$.

Observe that higher consumption tax, $(\tau_c \uparrow)$ has similar effect on the intra-temporal substitution. Higher tax on consumption means that with the same wage, the household can purchase less consumption. Leisure then becomes relatively cheaper than consumption, and the same substitution effect as described above occurs. Notice however that taxes on consumption also affect $R_{t+1}$, which is the inter-temporal rate of substitution. In particular, observe that if $\tau_{ct+1} > \tau_{ct}$, then $R_{t+1} \downarrow$. Lower real after-tax gross interest rate means that current consumption becomes cheaper relative to future consumption\footnote{Recall that real interest rate represents the price of current consumption in terms of future consumption. If the real interest rate is 5%, then consuming extra unit today, instead of saving it, means that you give up 1.05 units of future consumption.}. Thus, lower $(1 + \tau_{ct}) / (1 + \tau_{ct+1})$ makes the household want to consume more and save less. On top of that, as always, we have the general equilibrium effect which works through changes in equilibrium prices $r_t$. Again, without solving numerically a parametrized version of the model, we cannot be sure about the effects of $\tau_c$ on any of the equilibrium variables.

The same story is with $\tau_k$ and $\tau_x$. Higher taxes on capital income, reduce the real interest rate $R_{t+1}$ and thus make the current consumption cheaper. This force will, ceteris paribus, tend to increase consumption and reduce investment. But in general equilibrium, prices will be affected as well, so again we cannot predict exactly what would happen in this model economy. The important lesson here is that it is not easy to analyze the effects of fiscal policies even in a simple model, let alone in the real world. If someone tells you that they are confident about a certain tax reform, you should be very skeptical about it. In this model, we can subject the artificial economy to particular fiscal policy, and attempt to analyze the numerical results. We will be able to conclude which of the many effects dominated, and how the parameters of the model affect its response to the policies.

We can say a bit more about the permanent effects of fiscal policies, i.e., the effects of policies on the steady state. At the steady state, productivity and fiscal policy must be fixed over time: \( \{g_t, \tau_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty} = (g, \tau, \tau_c, \tau_x, \tau_w, \tau_k) \). The steady state conditions are rewritten again for convenience.

\[
\begin{align*}
[\text{Optimal labor}] & : \frac{u_2 (c, 1 - h)}{u_1 (c, 1 - h)} = \frac{(1 - \tau_w)}{(1 + \tau_c)} F_2 (k, h) \\
[\text{EE}] & : 1 = \beta \left[ \frac{1 - \tau_k}{1 + \tau_x} \right] F_1 (k, h) + 1 - \delta \\
[\text{Feas}] & : c + \delta k + g = F (k, h)
\end{align*}
\]

Notice that a fixed consumption tax $\tau_c$ does not affect the Euler equation. It does have effect on the intra-temporal substitution in the first optimality equation. Consumption becomes more expensive relative to leisure, so the substitution effect is to reduce consumption and to increase leisure (work less). Again, there is also income and general equilibrium effect. Higher consumption tax is accompanied by higher lump-sum transfers, which is a positive income effect. If households work less, then this is yet another general equilibrium effect through wages.
A bit more can be said about the effects of fiscal policies in a simplified version of this model with inelastic labor supply. In this version of the model, the equilibrium is characterized by only two equations

\[
EE: u'(c_t) = \beta u'(c_{t+1}) R_{t+1}
\]

where \( R_{t+1} = \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} \left[ \left( \frac{1 - \tau_{kt+1}}{1 + \tau_{xt}} \right) f'(k_{t+1}) + \left( \frac{1 + \tau_{xt+1}}{1 + \tau_{xt}} \right) (1 - \delta) \right] \)

\[
[Feas]: c_t + k_{t+1} + g_t = f(k_t) + (1 - \delta) k_t
\]

where \( f(k_t) \equiv F(k_t, 1) \). So in the absence of the intra-temporal condition, taxes on labor income are not distorting and work like lump-sum taxes. This is intuitive since the labor supply is always 1, so labor income is an exogenous constant from the household’s point of view. Consumption tax, when fixed, is not distorting as well. Notice that if \( \tau_{ct} = \tau_{ct+1} \forall t \), then regardless of the level of this tax, the steady state equations are unaffected by it.

\[
EE: 1 = \beta \left[ \frac{1 - \tau_k}{1 + \tau_x} f'(k) + 1 - \delta \right]
\]

\[
[Feas]: c + \delta k + g = f(k)
\]

So the effect of changes in consumption tax is temporary, and constant consumption tax has no effect on the steady state.

### 6.3 Quantitative Analysis of Taxes

Let us solve the NGM with taxes for a special case of Cobb-Douglas utility function and production function, i.e.

\[
u(c_t, 1 - h_t) = \alpha \ln(c_t) + (1 - \alpha) \ln(1 - h_t)\]

\[
F(k_t, h_t) = A_t k_t^\theta h_t^{1-\theta}
\]

In this case, the first order and feasibility equations become:

\[
[Optimal \ labor]: \left( \frac{1 - \alpha}{\alpha} \right) \frac{c_t}{1 - h_t} = \frac{(1 - \tau_{wt})}{(1 + \tau_{ct})} (1 - \theta) A_t k_t^\theta h_t^{1-\theta}
\]

\[
[EE]: \frac{c_{t+1} (1 + \tau_{xt})}{c_t (1 + \tau_{ct})} = \beta \left[ (1 - \tau_{kt+1}) \theta A_t k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + (1 + \tau_{xt+1}) (1 - \delta) \right] / (1 + \tau_{ct+1})
\]

\[
[Feas]: c_t + k_{t+1} + g_t = A_t k_t^\theta h_t^{1-\theta} + (1 - \delta) k_t
\]

And the steady state equations become:

\[
[Optimal \ labor]: \left( \frac{1 - \alpha}{\alpha} \right) \frac{c}{1 - h} = \frac{(1 - \tau_w)}{(1 + \tau_c)} (1 - \theta) Ak^\theta h^{-\theta}
\]

\[
[EE]: (1 + \tau_x) = \beta \left[ (1 - \tau_k) \theta Ak^{\theta-1} h^{1-\theta} + (1 + \tau_x) (1 - \delta) \right]
\]

\[
[Feas]: c + \delta k + g = Ak^\theta h^{1-\theta}
\]
The Matlab codes that solve this model are: (1) TDCEseq.m, (2) TDCEeq.m, and (3) TDCEsimulations.m. The first function contains the steady state equations, i.e., the above 3 equations with the unknowns \((c, h, k)\). The second function contains the equations for equilibrium time paths of \(\{c_t, h_t, k_t\}_{t=0}^{T}\). Remember that we approximate \(\infty\) with large enough time horizon \(T\), such that the model converges close enough to the steady state. The main script that the user needs to modify for performing different experiments is TDCEsimulations.m. This program solves for the steady state first, in order to determine the final point to which the model converges. Then the user defines which experiments she wants to perform, and runs the code. The program solves the model for the user provided fiscal policies.

Fiscal policy instruments are sequences of the form \(\{g_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty}\), where the lump-sum transfer is not one of them because we proved above that without borrowing or lending in this model, the government must balance its budget, so \(\tau_t\) adjusts accordingly. There are infinitely many policies that we can asses, but economists are usually interested in conducting particular types of experiments:

1. **Anticipated once-and-for-all change**, when some policy instrument, say consumption tax, changes once-and-for-all at a given date. For example, a sequence \(\tau_{ct} = 0 \ \forall t = 0, 1, \ldots, 8\) and \(\tau_{ct} = 20\% \ \forall t = 9, 10, \ldots\) is such experiment. Households are assumed to know this entire time path at period 0. Usually, we analyze the effect of a specific change in one policy instrument, so we can isolate the impact of particular fiscal policy.

2. **Anticipated one-time change**, when the policy instrument deviates from its steady state value only during one particular period, and the households know about this change in advance. For example, \(\tau_{ct} = 0\) for all \(t \neq 9\), and \(\tau_{ct} = 20\%\).

3. **Unanticipated once-and-for-all change**, when the policy instrument changes at period 0 from its steady state level, and stays there for ever. To conduct this experiment, we need to solve for steady state for some original level of fiscal policy instruments, and then make the desired change at period 0. For example, find the steady state for \(\{g_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty} = (0, 0, 0, 0, 0)\) and then solve for the time path where \(\tau_{ct} = 20\% \ \forall t = 0, 1, 2, \ldots\)

4. **Unanticipated one-time change**, when the policy instrument changes from its original level at time 0 only, and goes back to its original level after that. For example, start the economy at steady state that corresponds to \(\{g_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}_{t=0}^{\infty} = (0, 0, 0, 0, 0)\), and then solve for the time path where \(\tau_{ct} = 20\%\), and \(\tau_{ct} = 0 \ \forall t = 1, 2, \ldots\) The resulting time paths are called impulse response functions.

### 6.3.1 Example: anticipated once-and-for-all increase in labor tax

Suppose that the model parameters are \(\alpha = 0.45, \beta = 0.97, \delta = 0.05, \theta = 0.35, A = 1.\) The fiscal policy on the original steady state is given by \(g = 0.2\) and \(\tau_{ct} = \tau_{xt} = \tau_{wt} = \tau_{kt} = 0\). The experiment that we want to perform is a once-and-for-all increase in the tax on labor from the 10th period and on, i.e. \(\tau_{wt} = 0 \ \forall t = 0, 1, 2, \ldots, 9\) and \(\tau_{wt} = 20\% \ \forall t = 10, 11, \ldots\) Using the Matlab program **TDCEsimulations.m**, we generate the next figure:
Observe that since the change in policy is anticipated, the household changes its behavior even prior to the actual change in policy. The horizontal lines correspond to the original steady state with zero taxes. Notice that in anticipation of the labor tax increase the household reduces consumption in the very beginning, and accumulates capital. Also notice that the household works more than the steady state, in anticipation of the coming increase in labor taxes. The reason for this behavior is an attempt to smooth consumption. People with concave utility are risk averse and do not like abrupt changes from one period to the next. The way this household can smooth consumption is by investing more and accumulating capital prior to the date when he is hit by the labor taxes. Notice that after the taxes were imposed, the household gradually reduces his stock of capital. This smoothing of consumption is observed in the real world when students take loans, when workers save for retirement, and when people buy insurance. Risk aversion is the key factor that determines how much households will respond to different shocks. Degree of risk aversion is particularly important when we study people’s behavior during business cycles.

**Exercise 2** Notice that following the once-and-for-all increase in $\tau_w$, the prices $r_t$ and $w_t$ return to their original steady state in the long run. Prove that with CRS production function, once-and-for-all changes in $\tau_w$ or $\tau_c$, do not have a permanent effect on $r_t$ and $w_t$. That is, $r_t$ and $w_t$ converge to their original steady state.
Solution 2  Based on Euler’s Theorem of homogeneous functions, if the production function is h.d.1. (i.e. CRS), the partial derivatives (marginal products) are h.d.0. and depend only on the ratio of inputs. In particular,
\[ r = F_1(k, h) = F_1 \left( \frac{k}{h}, 1 \right) \]
\[ w = F_2(k, h) = F_2 \left( \frac{k}{h}, 1 \right) \]

The Euler Equation determines the steady state \( F_1(k, h) \),
\[ [EE] : 1 = \beta \left[ \left( \frac{1 - \tau_k}{1 + \tau_x} \right) F_1(k, h) + 1 - \delta \right] \]

The steady state \( F_1(k, h) = F_1 \left( \frac{k}{h}, 1 \right) \) determines the ratio \( k/h \), which in turn determines \( r \) and \( w \). Note that this ratio at steady state is affected only by \( \tau_k \) and \( \tau_x \). Thus, \( \tau_w \) or \( \tau_c \) have no effect on steady state \( k/h \), \( r \) and \( w \).

Exercise 3  Notice that following the once-and-for-all increase in \( \tau_w \), the level of capital stock permanently declines, and converges to a lower steady state level. Prove that with CRS production function, if higher \( \tau_w \) increases steady state leisure, then the steady state capital must decrease.

Solution 3  We proved in the previous exercise that an increase in \( \tau_w \) does not change the \( k/h \). Therefore, if leisure goes up in steady state, it means that \( h \) decreases, and therefore \( k \) must decrease by the same factor as \( h \), to keep the ratio \( k/h \) unchanged.

7  Risk Aversion

Most of the important decisions we face involve making choices under uncertainty. Decisions in the presence of uncertainty is a well developed field of economics, from which finance evolved. I can’t do justice to this field by writing a few pages in these notes, so I encourage people to learn more by reading a chapter about uncertainty in any graduate-level micro textbook. I will write here the very minimum that we need for this course.

Definition 5  A lottery is a probability distribution over some prizes.

For example
\[ L = \begin{cases} 
$1000 & \text{w.p. 0.5} \\
$500 & \text{w.p. 0.5} 
\end{cases} \]
is a lottery that pays $1000 or $500 with equal probabilities. More general example would be \( L = (\pi, x) \) such that \( \pi = \pi_1, \pi_2, \ldots, \pi_N \) is a vector of probabilities with \( \pi_i > 0 \) and \( \sum_{i=1}^N \pi_i = 1 \), and \( x = x_1, x_2, \ldots, x_N \) are the corresponding prizes so that you get a prize \( x_i \) with probability \( \pi_i \).

Whenever an individual needs to make a choice between uncertain alternatives, she chooses between different lotteries. Buying a car, pursuing an academic degree, adopting a pet or getting married, are all lotteries. Therefore, we need to define preferences over lotteries. Under some assumptions, consumer’s preferences over lotteries can be represented with expected utility.
Definition 6 Let $u(x)$ be utility function from non-random payoffs. Expected utility from a lottery with random prizes $x$ is $E[u(x)]$, where $E$ is the mathematical expectation of $u(x)$.

For example, suppose that utility of certain outcomes is $u(x_i)$ and outcome $i$ occurs with probability $\pi_i$. Then the expected utility from the lottery is

$$E[u(x)] = \sum_{i=1}^{N} \pi_i u(x_i)$$

As another example, consider the lottery that pays $1000 or $500 with equal probabilities. The expected utility from that lottery is

$$E[u(x)] = 0.5 \cdot u(1000) + 0.5 \cdot u(500)$$

Definition 7 An individual is risk averse if he prefers the expected value of any lottery to the lottery itself. If preferences are represented with expected utility, then risk aversion means $u[E(x)] > E[u(x)]$, utility from expected value of the lottery is greater than the expected utility from the lottery.

For example, consider the lottery that pays $1000 or $500 with equal probabilities. A risk averse person prefers the expected value of this lottery, $750 with certainty, to participating in the lottery, i.e.

$$u(750) > 0.5u(1000) + 0.5u(500)$$

Mathematically, the definition of risk aversion is equivalent to the function $u$ being strictly concave. Intuitively, a function is concave if a cord connecting two points on the graph of the function lies below the graph of the function. The right hand side of the last inequality is a point on a cord connecting two points on the graph of $u$, while the left hand side is a point on the graph of $u$. Intuitively, the more concave the utility function is, the more risk averse the individual is, so we expect that any measure of risk aversion should contain the term $u''(x)$.

Definition 8 Arrow-Pratt measure of relative risk aversion:

$$RRA = - \frac{u''(x)}{u'(x)}x$$

This function measures the extent to which the individual is willing to risk a certain fraction of his wealth, i.e. relative risk. For example, suppose that an investor is offered to invest a fraction of his wealth in an asset with risky return. It turns out that if the investor has higher $RRA$, he will invest a smaller fraction of wealth in any risky asset.

Since the utility is concave, $u''$ will always be negative, so the minus in front of the formula converts the measure to positive numbers. Thus, bigger $RRA$ means greater risk aversion. The division by $u'(x)$ makes the measure invariant to multiplication of $u$ by constant. For example, suppose one consumer has utility $u$ and another has $v = 5u$. These consumers are equally risk averse and will make the same choices between lotteries, but $v''$ will be 5 times greater than $u''$. So the division by the first derivative of $u$ makes the measure invariant with respect to multiplication by constant. The multiplication by $x$ delivers a measure of relative risk aversion.
**Example:** Calculate the A-P measure of relative risk aversion for

\[ u(x) = \begin{cases} \frac{x^{1-\gamma} - 1}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \ln(x) & \gamma = 1 \end{cases} \]

**Solution:**

\[ u'(x) = x^{-\gamma}, \quad u''(x) = -\gamma x^{-\gamma-1} \]

\[ RRA = -\frac{u''(x)}{u'(x)} x = -\frac{-\gamma x^{-\gamma-1}}{x^{-\gamma}} x = \gamma \]

Notice, that this utility function exhibits Constant Relative Risk Aversion (CRRA). This means that regardless of how big the wealth is, for given \( \gamma \) the individual has the same tendency to risk given fractions of his wealth. So for example, if Kim has \( \gamma = 0.5 \) and Ambrose has \( \gamma = 1 \), then Ambrose has higher relative risk aversion. In other words, he is less willing to risk a higher percentage of his wealth than Kim. The next figure shows the graphs of CRRA utility functions with different \( \gamma \).

![CRRA utility for various \( \gamma \)](image)

Notice, higher \( \gamma \) makes the graph more curved ("more concave").

### 7.1 Examples of choice under uncertainty

In this section we present several examples in which agents need to make choices while facing uncertainty.

#### 7.1.1 Demand for insurance

Suppose that Arnold has wealth \( w \) and faces a risk that with probability \( \pi \) his wealth will sustain damage \( d \in [0, w] \), and with probability \( 1 - \pi \) his wealth remains intact. Arnold can buy insurance with premium \( p \) per unit of wealth insured, and he needs to choose the amount of coverage \( q \) to purchase, where \( 0 \leq q \leq d \). Assuming that the utility over certain outcomes is represented by \( u(\cdot) \), which is strictly increasing and strictly concave, Arnold’s expected utility is

\[ E[u] = \pi u(w - d - pq + q) + (1 - \pi) u(w - pq) \]
Notice that in the case of damage occurring, he still needs to pay the premium, but he receives a compensation at the amount of his coverage \( q \). His problem is therefore

\[
\max_{0 \leq q \leq d} \pi u (w - d - pq + q) + (1 - \pi) u (w - pq)
\]

The first order condition for interior solution is:

\[
\pi u' (w - d + q^* (1 - p)) (1 - p) - (1 - \pi) u' (w - pq^*) p = 0
\]

The second order condition is

\[
\pi u'' (w - d + q^* (1 - p)) (1 - p)^2 + (1 - \pi) u'' (w - pq^*) p^2 < 0
\]

Thus, since Arnold is risk averse (utility is strictly concave i.e. \( u'' < 0 \)), the second order condition is satisfied. For interior solution, we must have

\[
\frac{\pi u' (w - d + q^* (1 - p)) (1 - p)}{u' (w - pq^*)} = \frac{(1 - \pi) u' (w - pq^*) p}{(1 - p) \pi}
\]

An important general result is that if insurance is actuarially fair, then a risk averse person will buy full coverage, i.e. \( q^* = d \). An actuarially fair insurance requires that \( p = \pi \), in which case the insurance company makes zero expected profit: \( pq - \pi q = 0 \). Thus

\[
\frac{u' (w - d + q^* (1 - p))}{u' (w - pq^*)} = 1
\]

\[
w - d + q^* (1 - p) = w - pq^*
\]

\[
q^* = d
\]

If \( p > \pi \), the insurance company makes positive profit and risk averse individuals will no buy full coverage, and possible will not buy any insurance at all if the premium is too high\(^8\). In order to find exactly how much insurance is purchased, we need to know the function \( u (\cdot) \).

### 7.1.2 Investment in risky asset

Suppose that Barney has a wealth \( w \) and can invest an amount \( x \in [0, w] \) in a risky asset. With probability \( \pi \) the net return on the asset is \(+r\) and with probability \( 1 - \pi \) the net return is \(-r\), where \( r > 0 \). Barney’s problem is therefore

\[
\max_{0 \leq x \leq w} \pi u (w + rx) + (1 - \pi) u (w - rx)
\]

The first order condition for interior optimum is

\[
\pi u' (w + rx^*) r - (1 - \pi) u' (w - rx^*) r = 0
\]

\(^8\)This might explain why many Americans do not purchase health insurance.
The second order condition for interior optimum:

\[ \pi u''(w + rx^*) r^2 + (1 - \pi) u''(w - rx^*) r^2 < 0 \]

which always holds for a risk averse person.

Will Barney invest any amount in the risky asset? Mathematically, having \( x^* > 0 \) is equivalent to having positive slope of expected utility at \( x = 0 \), as shown in the next figure.

Thus, the condition for \( x^* > 0 \) is

\[ \pi u'(w + r0) r - (1 - \pi) u'(w - r0) r > 0 \]
\[ \pi u'(w) r - (1 - \pi) u'(w) r > 0 \]
\[ \pi r - (1 - \pi) r > 0 \]
\[ \pi > \frac{1}{2} \]

This is another important and general result that any risk averse person will invest a positive amount in a risky asset as long as the expected return on that asset is positive (expected return is \( \pi r - (1 - \pi) r > 0 \)).

Suppose that Barney’s utility from certain outcomes is of the CRRA form. We will show that the higher is his risk aversion, the less he will invest in the risky asset. At the interior optimum we have

\[ \pi (w + rx^*)^{-\gamma} = (1 - \pi) (w - rx^*)^{-\gamma} \]
\[ \pi^{-1/\gamma} (w + rx^*) = (1 - \pi)^{-1/\gamma} (w - rx^*) \]
\[ w + rx^* = a(w - rx^*) \]
where \( a = \left( \frac{\pi}{1 - \pi} \right)^{1/\gamma} \)

Solving for \( x^* \)

\[ w + rx^* = aw - rax^* \]
\[ rax^* + rx^* = aw - w \]
\[ x^* = \frac{aw - w}{r(a + 1)} \]
\[ x^* = \left( \frac{a - 1}{a + 1} \right) \frac{1}{r} \]

(34)
The left hand side is the fraction of wealth invested in the risky asset. We can show that this fraction is decreasing in the Arrow-Pratt coefficient of relative risk aversion, $\gamma$:

$$\frac{\partial (x^*/w)}{\partial \gamma} = \frac{\partial (x^*/w)}{\partial a} \frac{\partial a}{\partial \gamma} = \left( \frac{a + 1 - (a - 1)}{(a + 1)^2} \right) \frac{\log \left( \frac{\pi}{1 - \pi} \right) \left( \frac{\pi}{1 - \pi} \right)^{1/\gamma} (-\gamma^{-2})}{\pi - (a - 1)}$$

$$= \left( \frac{2}{(1 + a)^2} \right) \frac{\log \left( \frac{\pi}{1 - \pi} \right) \left( \frac{\pi}{1 - \pi} \right)^{1/\gamma} (-\gamma^{-2})}{\pi - (a - 1)} < 0$$

Notice that since a risk averse person will invest in the risky asset only if $\pi > 1 - \pi$, we have $\log \left( \frac{\pi}{1 - \pi} \right) > 0$.

Notice also that the fraction invested in the risky asset is decreasing in $r$, according to (34). The reason is that higher $r$ means higher return in the good case, but also lower return in the bad case. Risk averse investor has concave utility, which is impacted more by losses than by gains of the same magnitude.

### 7.2 Elasticity of substitution

Risk aversion means that people prefer average over extremes, or in other words would like to smooth consumption. Our intuition tells us that in a dynamic model such as the NGM, risk averse households would prefer more-or-less equal consumption over time, as opposed to low consumption in one period and high consumption in the next period. Thus, intuitively, households willingness to substitute consumption in period $t$ and $t'$ should be related to their risk aversion. Formally, the willingness of households to substitute one good for another is measured by the elasticity of substitution between those two goods.

**Definition 9** Let the utility function be $u(x, y)$. The elasticity of substitution (in utility) between goods $x$ and $y$ is:

$$ES_{x,y} = \frac{\% \Delta (y/x)}{\% \Delta MRS_{x,y}} = \frac{\frac{d(y/x)}{dMRS_{x,y}}}{MRS_{x,y}}$$

In words, the elasticity of substitution measures how much the ratio of the two goods changes as the slope of the indifference curves changes. In the case of production function, the elasticity of substitution between two inputs measures how the ratio of the inputs changes as the slope of the isoquants changes. In that case, we replace $MRS$ with $TRS$.

**Example:** suppose that the household derives utility from consumption in two periods according to

$$u(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

with $\gamma > 0$, where it is understood that $\gamma = 1$ is the case of log utility. Find the elasticity of (inter-temporal) substitution between $c_1$ and $c_2$. First, find the marginal rate of (inter-temporal) substitution

$$u_1(c_1, c_2) = c_1^{-\gamma}, \quad u_2(c_1, c_2) = \beta c_2^{-\gamma}$$

$$\Rightarrow MRS_{c_1,c_2} = \frac{c_1^{1-\gamma}}{\beta c_2^{1-\gamma}} = \frac{1}{\beta} \left( \frac{c_2}{c_1} \right)$$

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Next, it is convenient to find $1/ES_{c_1,c_2}$:

$$
\frac{1}{ES_{c_1,c_2}} = \frac{dMRS_{c_1,c_2}}{d(c_2/c_1)} \cdot \frac{c_2/c_1}{MRS_{c_1,c_2}} = \frac{1}{\beta \gamma} \left( \frac{c_2}{c_1} \right)^{-1} \cdot \frac{c_2/c_1}{\frac{1}{\beta} \left( \frac{c_2}{c_1} \right)^{\gamma}} = \gamma
$$

The elasticity of substitution is therefore

$$ES_{c_1,c_2} = \frac{1}{\gamma}$$

This example illustrates that when the period utility is of the constant relative risk aversion (CRRA) form, the inter-temporal elasticity of substitution is also constant (CES), and is the inverse of the A-P measure of relative risk aversion. The next figure illustrates indifference curves with different parameter $\gamma$.

Notice that when $\gamma$ is large ($ES$ is low), the household is more risk averse and at the same time, the household is less willing to substitute inter-temporally. In the extreme when $\gamma \to \infty$, the indifference curves approach the perfect complements right angle curves. In that case, the household wants to preserve a constant ratio of $c_1/c_2$ and not willing to substitute inter-temporally ($ES \to 0$). In homework exercises you will experiment with different CES parameters to see how the household changes its response to fiscal policies, and to business cycles. The CES parameter is also the key to finding optimal policies for combating global warming. Any policy to reduce emissions will reduce current consumption but also may prevent a sharp drop in future consumption due to climatic disasters. Thus, if our $\gamma$ is high, the people are more risk averse and want to smooth consumption more. In that case, a policy that would prevent disastrous drops
in future consumption would be optimal. If on the other hand $\gamma$ is low, people are more tolerant towards risk, and don’t care about smoothing consumption. In that case, we don’t have to do anything about global warming, and wait for the disaster to hit. Our former faculty, Yanchun Zhang, is currently working on this topic.

8 Final Thoughts

Analyzing the effects of fiscal policies on the economy is not an easy task. We have made a step in the right direction by constructing a laboratory that helps us conduct controlled experiments with fiscal policies - a luxury that we don’t have with actual economies.

9 Appendix: Analytical Solution of Steady State

In this appendix, we solve the system of equations (15)-(17). Substituting (17) into (15) gives.

$$\left(\frac{1}{\alpha} - \frac{1}{h_{ss}}\right) \frac{A k_{ss}^\theta h_{ss}^{1-\theta} - \delta k_{ss}}{1 - h_{ss}} = (1 - \theta) A k_{ss}^\theta h_{ss}^{1-\theta}$$

Rearranging equation (16) gives the relationship between $h_{ss}$ and $k_{ss}$

$$1 = \beta \left[ \theta A h_{ss}^{\theta - 1} h_{ss}^{1-\theta} + 1 - \delta \right]$$

$$\frac{1}{\beta} - 1 + \delta = \theta A k_{ss}^{\theta - 1} h_{ss}^{1-\theta}$$

$$\left(\frac{1}{\beta} - 1 + \delta \right) k_{ss}^{1-\theta} = h_{ss}^{1-\theta}$$

$$h_{ss} = \eta k_{ss}$$

where $\eta = \left(\frac{1}{\beta} - 1 + \delta \right)^{\frac{1}{1-\theta}}$

Using $h_{ss} = \eta k_{ss}$ in equation (35) gives

$$\left(\frac{1}{\alpha} - \frac{1}{\eta k_{ss}}\right) \frac{A k_{ss}^\theta (\eta k_{ss})^{1-\theta} - \delta k_{ss}}{1 - \eta k_{ss}} = (1 - \theta) A k_{ss}^\theta (\eta k_{ss})^{1-\theta}$$

$$\left(\frac{1}{\alpha} - \frac{1}{\eta k_{ss}}\right) \frac{A\eta^{1-\theta} k_{ss} - \delta k_{ss}}{1 - \eta k_{ss}} = (1 - \theta) A\eta^{1-\theta}$$

$$k_{ss} \left(\frac{1}{\alpha} - \frac{1}{\eta k_{ss}}\right) \left(A\eta^{1-\theta} - \delta \right) = (1 - \theta) A\eta^{1-\theta} (1 - \eta k_{ss})$$

$$k_{ss} \left(\frac{1}{\alpha} - \frac{1}{\eta k_{ss}}\right) \left(A\eta^{1-\theta} - \delta \right) = (1 - \theta) A\eta^{1-\theta} - (1 - \theta) A\eta^{1-\theta} k_{ss}$$

$$k_{ss} \left(\frac{1}{\alpha} - \frac{1}{\eta k_{ss}}\right) \left(A\eta^{1-\theta} - \delta \right) + (1 - \theta) A\eta^{1-\theta} k_{ss} = (1 - \theta) A\eta^{1-\theta}$$

$$k_{ss} \left[\left(\frac{1}{\alpha} - \frac{1}{\eta k_{ss}}\right) \left(A\eta^{1-\theta} - \delta \right) + (1 - \theta) A\eta^{1-\theta}\right] = (1 - \theta) A\eta^{1-\theta}$$
Thus, the steady state equations are:

\[
\begin{align*}
    k_{ss} &= \frac{(1 - \theta) A \eta^{-\theta}}{\left[\left(\frac{1-\alpha}{\alpha}\right) (A\eta^{1-\theta} - \delta) + (1 - \theta) A\eta^{1-\theta}\right]} \\
h_{ss} &= \eta k_{ss} = \frac{(1 - \theta) A\eta^{1-\theta}}{\left[\left(\frac{1-\alpha}{\alpha}\right) (A\eta^{1-\theta} - \delta) + (1 - \theta) A\eta^{1-\theta}\right]} \\
c_{ss} &= Ak_{ss}^{\theta}h_{ss}^{1-\theta} - \delta k_{ss} = A\eta^{1-\theta}k_{ss} - \delta k_{ss} = (A\eta^{1-\theta} - \delta) k_{ss}
\end{align*}
\]  

(36)  

(37)  

(38)

where \( \eta = \left( \frac{\frac{1}{\beta} - 1 + \delta}{\theta A} \right)^{\frac{1}{1-\theta}} \)

The following proposition is left as an exercise.

**Proposition 10** The steady state labor supply and saving rate (or investment rate) are independent of the TFP.

**Proof.** I just provide a guideline for the proof. You need to work out the details in the homework. First, to show that \( h_{ss} \) is independent of \( A \), notice that \( A \) always appears multiplied by \( \eta^{1-\theta} \). Using the definition of \( \eta \), you need to show that \( A \) cancels out in the product \( A\eta^{1-\theta} \). The steady state saving (or investment) is \( s_{ss} = \delta k_{ss} \), from equation (17). The saving rate is therefore

\[
s = \frac{\delta k_{ss}}{Ak_{ss}^{\theta}h_{ss}^{1-\theta}}
\]

Using \( h_{ss} = \eta k_{ss} \) gives

\[
s = \frac{\delta}{A\eta^{1-\theta}}
\]

Which does not depend on \( A \). ■

The intuition is as follows. First, note that the labor supply depends on the demand for leisure. When \( A \uparrow \), the steady state wage \( w_{ss} \uparrow \). This has two effects on leisure - income effect and substitution effect. With higher income, consumers want to increase the consumption of normal goods, including leisure. But on the other hand, the price of leisure goes up when the wage is higher. It turns out that with Cobb-Douglas preferences, income and substitution effect cancel each other. Second, note that higher income makes consumers want to consume more now, as well as in the future. So it is not necessary that consumers will change the fraction of income consumed and saved. It turns out that with Cobb-Douglas preferences, consumers will not change their steady state saving rate.