1. Consider the Neoclassical Growth Model with government and taxes, discussed in class, and briefly described as follows. The economy consists of representative household, representative firm (owned by the household), and government, who live forever. The household’s lifetime utility function is $U (\{c_t, l_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u (c_t, l_t)$, where $c_t, l_t$ are consumption and leisure at time $t$, $u (\cdot, \cdot)$ is period utility function, and $\beta = \frac{1}{1 + \rho}$ is time discount factor ($\rho$ is the discount rate). The household has 1 unit of time, so the labor supply is $1 - l_t$. The household owns the capital stock $k_t$, with the law of motion $k_{t+1} = (1 - \delta) k_t + x_t$, where $x_t$ is investment and $k_0 > 0$ is given. The household earns a pre-tax real wage $w_t$ per unit of labor supplied to the firm, a rental rate $r_t$ per unit of capital stock rented to the firm, and $\pi_t$ the profit (dividend) from the representative firm. There is a single representative firm that produces the output in this economy, with production function $Y_t = F (k_t, L_t)$, where $K_t$ is capital and $L_t$ is labor. Assume that $F (\cdot, \cdot)$ satisfies all the assumptions of a neoclassical production function. The government imposes the following taxes: $\{\tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt}\}$ on consumption, investment, labor income and capital income. The government uses the taxes to finance its expenditures $g_t$ and lump-sum transfers $t_t$. Thus, in each period, the government’s income is $\tau_{ct} c_t + \tau_{xt} x_t + \tau_{wt} w_t h_t + \tau_{kt} r_t k_t$ and its outlays are $g_t + \tau_t$. For simplicity, we will assume that the government always balances its budget:

$$g_t + \tau_t = \tau_{ct} c_t + \tau_{xt} x_t + \tau_{wt} w_t h_t + \tau_{kt} r_t k_t \quad \forall t$$

The economy is closed, so the feasibility constraint is

$$c_t + x_t + g_t = Y_t$$

(a) Solve for equilibrium conditions:

[Optimal labor] : $\frac{u_2 (c_t, 1 - h_t)}{u_1 (c_t, 1 - h_t)} = \frac{(1 - \tau_{wt})}{(1 + \tau_{ct})} F_2 (k_t, h_t)$

[EE] : $u_1 (c_t, 1 - h_t) = \beta u_1 (c_{t+1}, 1 - h_{t+1}) R_{t+1}$

[Feas] : $c_t + k_{t+1} + g_t = F (k_t, h_t) + (1 - \delta) k_t$

where

$$R_{t+1} = \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} \left[ \left( \frac{1 - \tau_{kt+1}}{1 + \tau_{xt}} \right) F_1 (k_{t+1}, h_{t+1}) + \left( \frac{1 + \tau_{xt+1}}{1 + \tau_{xt}} \right) (1 - \delta) \right]$$

1
The household’s problem is:

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u (c_t, 1 - h_t)$$

s.t.

$$(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) x_t = (1 - \tau_{wt}) w_t h_t + (1 - \tau_{kt}) r_t k_t + \tau_t + \pi_t$$

$$k_{t+1} = (1 - \delta) k_t + x_t$$

Plugging the law of motion of capital into the budget constraint, simplifies the household’s problem to

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u (c_t, 1 - h_t)$$

s.t.

$$(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) k_{t+1} = (1 - \tau_{wt}) w_t h_t + (1 - \tau_{kt}) r_t k_t + (1 + \tau_{xt}) (1 - \delta) k_t + \tau_t$$

The Lagrange function is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u (c_t, 1 - h_t) - \sum_{t=0}^{\infty} \lambda_t [(1 + \tau_{ct}) c_t + (1 + \tau_{xt}) k_{t+1}$$

$$- (1 - \tau_{wt}) w_t h_t - (1 - \tau_{kt}) r_t k_t - (1 + \tau_{xt}) (1 - \delta) k_t - \tau_t]$$

The first order necessary conditions with respect to $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ are

$$[c_t] : \beta^t u_1 (c_t, 1 - h_t) - \lambda_t (1 + \tau_{ct}) = 0$$

$$[h_t] : -\beta^t u_2 (c_t, 1 - h_t) + \lambda_t (1 - \tau_{wt}) w_t = 0$$

$$[k_{t+1}] : -\lambda_t (1 + \tau_{xt}) + \lambda_{t+1} [(1 - \tau_{kt+1}) r_{t+1} + (1 + \tau_{xt+1}) (1 - \delta)] = 0$$

Rearranging these conditions yields the intra-temporal and inter-temporal optimality conditions

$$\frac{u_2 (c_t, 1 - h_t)}{u_1 (c_t, 1 - h_t)} = \frac{(1 - \tau_{wt})}{(1 + \tau_{ct})} w_t$$

$$\frac{u_1 (c_t, 1 - h_t) (1 + \tau_{xt})}{(1 + \tau_{ct})} = \beta \frac{u_1 (c_{t+1}, 1 - h_{t+1})}{(1 + \tau_{ct+1})} [(1 - \tau_{kt+1}) r_{t+1} + (1 + \tau_{xt+1}) (1 - \delta)]$$

Combining these with firm’s profit maximizing conditions, $r_t = F_1 (k_t, h_t), w_t = F_2 (k_t, h_t)$, market clearing $h_t = L_t$, $k_t = K_t$, and feasibility, gives the equilibrium conditions for $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$:

$$[\text{Optimal labor}] : \frac{u_2 (c_t, 1 - h_t)}{u_1 (c_t, 1 - h_t)} = \frac{(1 - \tau_{wt})}{(1 + \tau_{ct})} F_2 (k_t, h_t)$$

$$[\text{EE}] : u_1 (c_t, 1 - h_t) = \beta u_1 (c_{t+1}, 1 - h_{t+1}) R_{t+1}$$

$$[\text{Feas}] : c_t + k_{t+1} + g_t = F (k_t, h_t) + (1 - \delta) k_t$$
where
\[ R_{t+1} = \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} \left[ \left( \frac{1 - \tau_{kt+1}}{1 + \tau_{xt}} \right) F_1(k_{t+1}, h_{t+1}) + \left( \frac{1 + \tau_{xt+1}}{1 + \tau_{xt}} \right) (1 - \delta) \right] \]

(b) Show that \( R_{t+1} \) is the relative price of \( c_t \) in terms of \( c_{t+1} \). Hint: increase \( c_t \) by 1 unit, and show that the household must give up \( R_{t+1} \) units of \( c_{t+1} \).

Consuming 1 more unit of \( c_t \) cost \( 1 + \tau_{ct} \) units of the final good. If instead that amount was invested, the household could buy \( \frac{1 + \tau_{ct}}{1 + \tau_{xt}} \) units of physical capital. In the next period, this capital could generate a return (after tax) of \( (1 - \tau_{kt+1}) r_{t+1} \frac{1 + \tau_{ct}}{1 + \tau_{xt}}, \) plus the (after tax) value of the non-depreciated capital \( (1 + \tau_{xt+1}) (1 - \delta) \frac{1 + \tau_{ct}}{1 + \tau_{xt}} \), with total return on investment of
\[
\left[ (1 - \tau_{kt+1}) r_{t+1} + (1 + \tau_{xt+1}) (1 - \delta) \right] \frac{1 + \tau_{ct}}{1 + \tau_{xt}}
\]

This amount can be spent to buy consumption in period \( t + 1 \) at the price of \( 1 + \tau_{ct+1} \):
\[
\Delta c_{t+1} = \frac{1}{1 + \tau_{ct+1}} \left[ (1 - \tau_{kt+1}) r_{t+1} + (1 + \tau_{xt+1}) (1 - \delta) \right] \frac{1 + \tau_{ct}}{1 + \tau_{xt}}
\]

Thus, by consuming 1 extra unit of \( c_t \) we are giving up \( R_{t+1} \) units of \( c_{t+1} \), so \( R_{t+1} \) is the relative price of current consumption \( (c_t) \) in units of future consumption \( (c_{t+1}) \).

(c) Interpret the Euler Equation (left hand side and right hand side) written as follows:
\[
\frac{u_1(c_t, 1 - h_t)}{\beta u_1(c_{t+1}, 1 - h_{t+1})} = R_{t+1}
\]

The left hand side is the Marginal Rate of Substitution between \( c_t \) and \( c_{t+1} \), while the right hand side is the relative price of \( c_t \) in units of \( c_{t+1} \). That is, the above Euler Equation means
\[
MRS_{ct,ct+1} = \frac{pc_t}{pc_{t+1}}
\]

(d) We say that taxes not-distorting, if the competitive equilibrium allocation coincides with the solution to the social planner’s problem below (i.e., the competitive equilibrium allocation is Pareto Optimal). Otherwise, we say that the taxes are distorting.

\[
\max_{\{c_t, h_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)
\]
\[\text{s.t.}\]
\[
c_t + k_{t+1} + g_t = F(k_t, h_t) + (1 - \delta) k_t
\]
\[
k_0 > 0 \text{ given}
\]
Derive the conditions for Pareto Optimal allocation \( \{ c_t, h_t, k_t \}_{t=0}^{\infty} \).

The Lagrange function for the Social Planner’s problem is:

\[
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) - \sum_{t=0}^{\infty} \lambda_t \left[ c_t + k_{t+1} + g_t - F(k_t, h_t) - (1 - \delta) k_t \right]
\]

The first order necessary conditions with respect to \( \{ c_t, h_t, k_{t+1} \}_{t=0}^{\infty} \) are

\[
\begin{align*}
[ c_t ] & : \beta^t u_1(c_t, 1 - h_t) - \lambda_t = 0 \\
[ h_t ] & : -\beta^t u_2(c_t, 1 - h_t) + \lambda_t F_2(k_t, h_t) = 0 \\
[ k_{t+1} ] & : -\lambda_t + \lambda_{t+1} \left[ F_1(k_t, h_t) + 1 - \delta \right] = 0
\end{align*}
\]

Rearranging and combining with feasibility, gives the conditions for Pareto Optimal allocation \( \{ c_t, h_t, k_t \}_{t=0}^{\infty} \):

\[
\begin{align*}
[ \text{Optimal labor} ] & : \frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = F_2(k_t, h_t) \\
[ \text{EE} ] & : u_1(c_t, 1 - h_t) = \beta u_1(c_{t+1}, 1 - h_{t+1}) \left[ F_1(k_t, h_t) + 1 - \delta \right] \\
[ \text{Feas} ] & : c_t + k_{t+1} + g_t = F(k_t, h_t) + (1 - \delta) k_t
\end{align*}
\]

(e) Which taxes are distorting and which ones are not?

Distorting taxes are \( \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt} \) because they prevent the competitive equilibrium allocation from being Pareto Optimal. The only non-distorting tax is the lump-sum transfer, \( \tau_t \).

(f) In order to see whether tax policy changes have a permanent, or only temporary effect on the competitive equilibrium, we need to look at the steady state equilibrium. Write the steady equilibrium conditions for fixed fiscal policy \( \{ g_t, \tau_t, \tau_{ct}, \tau_{xt}, \tau_{wt}, \tau_{kt} \}^{\infty}_{t=0} = (g, \tau, \tau_c, \tau_x, \tau_w, \tau_k) \) and productivity.

\[
\begin{align*}
[ \text{Optimal labor} ] & : \frac{u_2(c, 1 - h)}{u_1(c, 1 - h)} = \frac{(1 - \tau_w)}{(1 + \tau_c)} F_2(k, h) \\
[ \text{EE} ] & : 1 = \beta \left[ \left( \frac{1 - \tau_k}{1 + \tau_x} \right) F_1(k, h) + 1 - \delta \right] \\
[ \text{Feas} ] & : c + \delta k + g = F(k, h)
\end{align*}
\]

(g) Prove that steady state level of investment rate is affected only by \( \tau_k \) and \( \tau_x \), but not affected by other taxes. It is easier to prove this proposition first for Cobb-Douglas production, and then generalize to any Neoclassical production function.

The steady state level of investment \( \delta k \), and investment rate is \( \delta k / F(k, h) \). In the Cobb-Douglas case, the investment rate is

\[
\frac{\delta k}{F(k, h)} = \frac{\delta k}{A k^\theta h^{1-\theta}} = \frac{\delta}{A k^{\theta-1} h^{1-\theta}}
\]
Notice that the Euler Equation with Cobb-Douglas production is

\[ 1 = \beta \left[ \left( \frac{1 - \tau_k}{1 + \tau_x} \right) \theta A k^{\theta - 1} h^{1 - \theta} + 1 - \delta \right] \]

We can see that the term \( A k^{\theta - 1} h^{1 - \theta} \) is affected only by \( \tau_k \) and \( \tau_x \), and not by any other taxes, and therefore investment rate, \( \delta/A k^{\theta - 1} h^{1 - \theta} \), is also affected only by these two taxes.

With general Neoclassical production function, we know that \( F_1(k, h) = F_1(1, \frac{h}{k}) \). This follows from \( F \) being homogeneous of degree 1 (CRS), and by Euler’s theorem the partial derivatives of \( F \) are homogeneous of degree zero. Thus, the Euler equation can be written as

\[ 1 = \beta \left[ \left( \frac{1 - \tau_k}{1 + \tau_x} \right) F_1 \left( 1, \frac{h}{k} \right) + 1 - \delta \right] \]

and we see that \( \frac{h}{k} \) is affected only by \( \tau_k \) and \( \tau_x \), and not by any other taxes. The investment rate can be shown to depend only on the ratio \( h/k \) (labor/capital).

\[ i = \frac{\delta k}{F(k, h)} = \frac{\delta}{\frac{1}{k} F(k, h)} = \frac{\delta}{F \left( 1, \frac{h}{k} \right)} \]

The last step used the definition of constant returns to scale.

Next, we prove the theorem that was used in this question.

**Proposition 1 (Euler Theorem for Homogeneous Functions).** Let \( F(x_1, x_2, ..., x_m) \) be homogeneous of degree \( n \), i.e.

\[ F(\lambda x_1, \lambda x_2, ..., \lambda x_m) = \lambda^n F(x_1, x_2, ..., x_m) \quad \forall \lambda > 0 \]

Then, the partial derivatives of \( F \) are homogeneous of degree \( n - 1 \).

**Proof.** Differentiate the definition with respect to \( x_i \), \( i = 1, 2, ..., m \):

\[ F_i(\lambda x_1, \lambda x_2, ..., \lambda x_m) \lambda = \lambda^n F_i(x_1, x_2, ..., x_m) \]
\[ F_i(\lambda x_1, \lambda x_2, ..., \lambda x_m) = \lambda^{n-1} F_i(x_1, x_2, ..., x_m) \]

The last equation is the definition of \( F_i \) being homogeneous of degree \( n - 1 \). Thus, in a special case of CRS (\( n = 1 \)), the partial derivatives are homogeneous of degree zero. ■

(h) Prove that investment rate at steady state is decreasing in \( \tau_k \) and in \( \tau_x \).

Applying the Euler theorem, we can write the Euler Equation as:

\[ 1 = \beta \left[ \left( \frac{1 - \tau_k}{1 + \tau_x} \right) F_1 \left( \frac{k}{h}, 1 \right) + 1 - \delta \right] \]
Notice that \( F_1 (\frac{k}{h}, 1) \) is decreasing function in \( \frac{k}{h} \), by properties of Neoclassical production function (marginal product is positive and diminishing, i.e. \( F_1 (k, h) > 0, F_{11} (k, h) < 0 \)). Rearranging that Euler Equation, gives

\[
1 + \rho = \left( \frac{1 - \tau_k}{1 + \tau_x} \right) F_1 \left( \frac{k}{h}, 1 \right) + 1 - \delta
\]

\[
(\rho + \delta) \left( \frac{1 + \tau_x}{1 - \tau_k} \right) = F_1 \left( \frac{k}{h}, 1 \right)
\]

Thus, higher \( \tau_k \) or \( \tau_x \) increase the marginal product of capital, and lower the capital/labor ratio \( k/h \) (increase \( h/k \)). In the previous section we proved that investment rate is decreasing in \( h/k \). To summarize the proof,

\[
\tau_k \uparrow, \tau_x \uparrow \implies F_1 \left( \frac{k}{h}, 1 \right) \uparrow \implies \left( \frac{k}{h} \right) \downarrow, \left( \frac{h}{k} \right) \downarrow \implies \frac{\delta}{F (1, \frac{h}{k})} \downarrow
\]

The first two steps can be illustrated graphically as follows:

2. Consider a special case of the Neoclassical Growth Model with government and taxes, and with inelastic labor supply. That is, the period utility function is \( u(c_t) \).

(a) Write the conditions for competitive equilibrium for this economy.

In this economy, in equilibrium we must have \( h_t = 1 \ \forall t \), since the household does not care about leisure and always supplies the 1 unit of time. The equilibrium output is therefore \( F (k_t, 1) \equiv f (k_t) \). Thus, the equilibrium conditions do not include the optimal labor choice, and are as follows:

\[
[EE] : u' (c_t) = \beta u' (c_{t+1}) R_{t+1}
\]

\[
[Feas] : c_t + k_{t+1} + g_t = f (k_t) + (1 - \delta) k_t
\]
where

\[ R_{t+1} = \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} \left[ \left( \frac{1 - \tau_{kt+1}}{1 + \tau_{kt}} \right) f'(k_{t+1}) + \left( \frac{1 + \tau_{xt+1}}{1 + \tau_{xt}} \right) (1 - \delta) \right] \]

(b) Which taxes are distorting and which ones are not?

Distorting taxes are \( \tau_{ct}, \tau_{xt}, \tau_{kt} \), because they prevent the competitive equilibrium allocation from being Pareto Optimal. The non-distorting taxes are \( \tau_{wt}, \tau_{t} \).

(c) Write the conditions for steady state equilibrium with constant fiscal policies and constant productivity.

\[
\begin{align*}
[\text{EE}] & : 1 = \beta \left[ \left( \frac{1 - \tau_k}{1 + \tau_x} \right) f'(k) + (1 - \delta) \right] \\
[\text{Feas}] & : c + \delta k + g = f(k)
\end{align*}
\]

(d) Prove that an increase in taxes on investment or capital income reduces the steady state capital.

The proof is identical to question 1-h, with \( k \) instead of \( k/h \).

(e) Which taxes are non-distorting in the long run (do not affect the steady state)?

The non-distorting taxes are \( \tau_{c}, \tau_{w} \) and \( \tau \). The other two taxes are distorting: \( \tau_{x}, \tau_{k} \).
3. Consider the Neoclassical Growth Model with taxes from question 1, with \( u(c_t, 1 - h_t) = \alpha \ln(c_t) + (1 - \alpha) \ln(1 - h_t) \), and \( F(k_t, h_t) = A_t k_t^{\delta} h_t^{1 - \delta} \). For simulating the effects of fiscal policies in this model, use the Matlab program TDCEsimulations.m. For this question, set the parameters \( \alpha = 0.45, \beta = 0.97, \delta = 0.05, \theta = 0.35 \) and \( A_{ss} = 1, g_{ss} = 0.2, (\tau_c, \tau_x, \tau_w, \tau_k)_{ss} = (0, 0, 0, 0) \).

(a) Perform the experiment where nothing ever changes, i.e. the economy starts at steady state, and none of the fiscal policy instruments ever change. Present the resulting time paths.

(b) Preform the experiment of anticipated increase in taxes on labor income, from period 10 onward: \( \tau_{w}(10:end) = \tau_{w, ss} + 0.2 \). Present the resulting time paths.

(c) What was the effect of higher labor taxes on household’s labor supply? Was the effect temporary or permanent?
Higher labor taxes decreased the steady state labor supply - permanent effect. In the short run, the labor supply increased, in anticipation of higher future labor tax.

(d) Provide economic intuition for the result in the last section.

Higher labor tax makes leisure cheaper, relative to consumption. Thus, the direct effect of higher labor taxes is to consume more leisure, and work less. In addition, since government spending did not change, the household receives higher lump-sum transfer. This positive income effect increases leisure even more (normal good), and lowers labor supply. Notice that in the short run the opposite happens - the labor supply increased, in anticipation of higher future labor tax.

(e) Is there evidence of consumption smoothing in your graphs? Explain.

There is evidence of consumption smoothing. The household reduces consumption before period 10, when higher tax on labor starts. Once the tax is enacted, consumption declines gradually.
(f) What is the mechanism that the household uses to smooth consumption?

The household accumulates physical capital, in anticipation of higher labor tax, and once the tax is enacted, the household consumes the accumulated capital.

4. Consider the Neoclassical Growth Model with taxes and inelastic labor supply from question 2, with \( u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} \), where \( \gamma > 0, \gamma \neq 1 \) and \( u(c_t) = \ln(c_t) \) when \( \gamma = 1 \). The production function is \( F(k_t, h_t) = A_t k_t^\theta h_t^{1-\theta} \). For simulating the effects of fiscal policies in this model, use the Matlab program `TDCEsimulations_simple.m`. The initial steady state parameters are: \( \gamma = 2, \beta = 0.95, \delta = 0.2, \theta = 0.35 \) and \( A_{ss} = 1, g_{ss} = 0.2, (\tau_c, \tau_x, \tau_w, \tau_k)_{ss} = (0, 0, 0, 0) \).

(a) Perform the experiment where nothing ever changes, i.e. the economy starts at steady state, and none of the fiscal policy instruments ever change. Present the resulting time paths.
(b) Report the steady state level of consumption, labor supply, capital, output and wages.

\[
\begin{array}{cccc}
 c_{ss} & h_{ss} & k_{ss} & y_{ss} \\
 0.6426 & 1 & 1.4900 & 1.1406 \\
 w_{ss} & & & 0.7642
\end{array}
\]

(c) Perform the experiment of anticipated increase in government spending from 0.2 to 0.4, starting from period 10 onward: \( g(10:\text{end}) = g_{ss} + 0.2 \). Present the resulting time paths.

(d) Explain why the consumption drops even prior to the change in fiscal policy?

Consumption smoothing.

(e) How is the household able to smooth consumption in anticipation of the fiscal change?

The household accumulates physical capital, in anticipation of higher labor tax, and once the tax is enacted, the household consumes the accumulated capital.
(f) Perform the same experiment of doubling the government spending from the 10th period on, but now change the CES parameter to $\gamma = 0.02$. Present the resulting time paths.

(g) Explain why there is much less consumption smoothing when $\gamma = 0.02$ than when $\gamma = 2$. (Hint: recall the relationship between risk aversion and the willingness of the household to substitute inter-temporally).

Smaller $\gamma$ means that the household is less risk averse and can tolerate sudden changes in consumption. Put differently, the inter-temporal elasticity of substitution is higher when $\gamma$ is smaller, which means that the household is more willing to substitute inter-temporally and does not want to smooth consumption as much as a more risk averse household.

5. Suppose that Nick has wealth $w$ and faces a risk that with probability $\pi$ his wealth will sustain damage $d \in [0, w]$, and with probability $1 - \pi$ his wealth remains intact. Nick can buy insurance with premium $p$ per unit of wealth insured, and he needs to
choose the amount of coverage \( q \) to purchase, where \( 0 \leq q \leq d \). Assuming that the utility over certain outcomes is represented by \( u(\cdot) \), which is strictly increasing and strictly concave.

(a) Write Nick’s expected utility when he chooses to insure an amount \( q \).

With probability \( \pi \) Nick’s wealth is \( w - d - pq + q \) (wealth sustains damage \( d \), gets coverage \( q \), and still pays premium \( pq \)), and with probability \( 1 - \pi \) his wealth is \( w - pq \) (no damage, only pays premium). Thus, the optimal coverage problem is

\[
\max_{0 \leq q \leq d} \pi u(w - d - pq + q) + (1 - \pi) u(w - pq)
\]

(b) Write the first order necessary condition for optimal \( q \).

The first order condition for interior solution is:

\[
\pi u'(w - d + q^*(1 - p))(1 - p) - (1 - \pi) u'(w - pq^*) p = 0
\]

(c) Write the second order sufficient condition for global maximum and prove that it is satisfied in this problem.

The second order condition is

\[
\pi u''(w - d + q^*(1 - p))(1 - p)^2 + (1 - \pi) u''(w - pq^*) p^2 < 0
\]

Thus, since the individual is risk averse (utility is strictly concave i.e. \( u'' < 0 \)), the second order condition is satisfied.

(d) Prove that if insurance is actuarially fair, then a risk averse person will buy full coverage, i.e. \( q^* = d \). An actuarially fair insurance requires that \( p = \pi \), in which case the insurance company makes zero expected profit: \( pq - \pi q = 0 \).

An actuarially fair insurance requires that \( p = \pi \), in which case the insurance company makes zero expected profit: \( E(\Pi) = pq - \pi q = 0 \). Thus

\[
\pi u'(w - d + q^*(1 - p))(1 - \pi) - (1 - \pi) u'(w - pq^*) \pi = 0
\]
\[
u'(w - d + q^*(1 - p)) - u'(w - pq^*) = 0
\]
\[
\frac{u'(w - d + q^*(1 - p))}{u'(w - pq^*)} = 1
\]
\[
w - d + q^*(1 - p) = w - pq^*
\]
\[
q^* = d
\]

6. Suppose Oscar has wealth \( w \) and his preferences over risky alternatives are described by Expected Utility Theory, and his utility over certain outcomes is represented by \( u(\cdot) \), which is strictly increasing and strictly concave. He can divide his wealth between investment in risky asset, with random return \( r \), and a risk-free asset with guaranteed return \( r_f \). Let the amount invested in risky asset be \( x \in [0, w] \).
(a) Write Oscar’s optimal investment problem.

The amount $x$ is invested in risky asset with return $r$ and the rest $w - x$ is invested in risk-free asset with return $r_f$. Thus, the future wealth is $x(1 + r) + (w - x)(1 + r_f) = w(1 + r_f) + x(r - r_f)$. The optimal investment problem is therefore:

$$\max_{0 \leq x \leq w} E[u(w(1 + r_f) + x(r - r_f))]$$

(b) Write the first order necessary condition for optimal $x$.

The first order condition for interior optimum, $x^*$, is

$$E[u'(w(1 + r_f) + x^*(r - r_f))(r - r_f)] = 0$$

(c) Write the second order sufficient condition for global maximum and prove that it is satisfied in this problem.

$$E[u''(w(1 + r_f) + x(r - r_f))(r - r_f)^2] < 0, \forall x$$

The above condition holds for all $x$ if Oscar is risk-averse ($u(\cdot)$ is strictly concave).

(d) Prove that Oscar will invest positive amount in risky asset ($x^* > 0$) if and only if the expected return on the risky asset is greater than the risk-free return: $E(r) > r_f$.

The condition of positive investment in risky asset, $x^* > 0$, is (slope of utility at $x = 0$ is positive):

$$E[u'(w(1 + r_f))(r - r_f)] > 0$$

$$u'(w(1 + r_f))E(r - r_f) > 0$$

Which holds if and only if

$$E(r) > r_f$$