Final exam

Wednesday, May 18

1 hours, 30 minutes

Name: ___________________________________

Instructions

1. This is closed book, closed notes exam.
2. No calculators of any kind are allowed.
3. Show all the calculations.
4. If you need more space, use the back of the page.
5. Fully label all graphs.
6. Derive means that you need to show steps
7. Write means that you don’t need to show steps, just write the result.

Good Luck 😊
1. (30 points). Consider the Neoclassical Growth Model discussed in class. There is a single representative household and a single representative firm, that live forever. The household’s period utility function \( u(c_t, l_t) \), and the lifetime utility is
\[
U(\{c_t, l_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),
\]
where \( 0 < \beta < 1 \) is the discount factor. The household has 1 unit of time, so the labor supply is \( h_t = 1 - l_t \). The household owns the capital stock \( k_t \), with the law of motion \( k_{t+1} = (1 - \delta)k_t + x_t \), where \( x_t \) is investment and \( k_0 > 0 \) is given. The household receives a real wage \( w_t \) per unit of labor supplied to the firm, a rental rate \( r_t \) per unit of capital stock rented to the firm, and \( \pi_t \) - the profit (dividend) from the representative firm. There is a single representative firm that produces the output in this economy, with production function \( y_t = F(K_t, L_t) \), where \( K_t \) is capital and the \( L_t \) is labor. Assume that \( F(K_t, L_t) \) satisfies all the assumptions of a neoclassical production function. The economy is closed and there is no government, thus the feasibility constraint is:
\[
c_t + x_t = y_t.
\]

a. (10 points). Derive the sufficient conditions for equilibrium sequences \( \{c_t, h_t, k_{t+1}\}_{t=0}^{\infty} \), i.e. \( \forall t = 0, 1, 2, \ldots \)

\[
(1) \quad \frac{u_2(c_t, l_t)}{u_1(c_t, l_t)} = F_2(k_t, h_t)
\]

\[
(2) \quad u_1(c_t, l_t) = \beta u_1(c_{t+1}, l_{t+1})[F_1(k_{t+1}, h_{t+1}) + 1 - \delta]
\]

\[
(3) \quad c_t + x_t = F(k_t, h_t) + (1 - \delta)k_t
\]

Explain your steps clearly.

Household’s problem:
\[
\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)
\]
subject to:
\[
c_t + k_{t+1} = w_t h_t + r_t k_t + (1 - \delta)k_t + \pi_t
\]

Notice that the profit is zero since the production function has constant returns to scale. Also notice that we substituted the leisure from the time constraint: \( l_t = 1 - h_t \), and the choice variable now is \( h_t \) - the labor supplied (work time).
Lagrange function:

\[ L = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) - \sum_{t=0}^{\infty} \lambda_t \left[ c_t + k_{t+1} - w_t h_t - r_t k_t - (1-\delta)k_t \right] \]

First order conditions:

\[ [c_t]: \beta^t u_1(c_t, 1-h_t) - \lambda_t = 0 \]
\[ [h_t]: -\beta^t u_2(c_t, 1-h_t) + \lambda_t w_t = 0 \]
\[ [k_{t+1}]: -\lambda_t + \lambda_{t+1} [r_{t+1} + 1-\delta] = 0 \]

Combining the conditions for \( c_t \) and \( h_t \), gives:

\[ \frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = w_t \]

Using the condition for consumption at time \( t \) and \( t+1 \) in \([ k_{t+1} ]\):

\[ u_1(c_t, 1-h_t) = \beta u_1(c_{t+1}, 1-h_{t+1}) [r_{t+1} + 1-\delta] \]

**Firm’s problem:**

\[ \max_{K_t, L_t} \pi_t = F(K_t, L_t) - w_t L_t - r_t K_t \]

First order conditions:

\[ w_t = F_2(K_t, L_t) = F_2(k_t, h_t) \]
\[ r_{t+1} = F_2(K_{t+1}, L_{t+1}) = F_2(k_{t+1}, h_{t+1}) \]

The last step substituted the market clearing conditions (\( k_t = K_t, \ h_t = L_t \)). Combining these prices with the household conditions, gives equilibrium equations (1) and (2).

Combining the feasibility constraint with the law of motion of capital:

\[ c_t + x_t = y_t \]
\[ k_{t+1} = (1-\delta)k_t + x_t \]

This gives feasibility constraint (3).
b. (10 points). Provide economic intuition for the optimal investment condition (Euler equation):

$$u_1(c_t,1-h_t) = \beta u_1(c_{t+1},1-h_{t+1})[r_{t+1} + 1-\delta]$$

The left hand side is the “pain” (decline in utility) as a result of investing extra unit of income in physical capital (and therefore giving up 1 unit of consumption) in period $t$. Recall that the marginal utility of consumption is the change in utility resulting from 1 unit change in consumption.

The right hand side is the utility gain from that investment. In period $t + 1$, the return on this investment, in units of consumption, is equal to $r_{t+1} +$ the non-depreciated unit of capital originally created. To convert this return into utility we multiply by the marginal utility from consumption, and to convert to present value we multiply by the discount factor.

Thus, the optimal investment condition requires balancing the marginal pain and the marginal gain from investment. Any model of investment must have a condition similar to this one.
c. (10 points). Prove that the competitive equilibrium allocation 
\( \{c_t, h_t, k_{t+1}\}_{t=0}^{\infty} \) is Pareto Optimal. That is, prove that the competitive 
equilibrium allocation solves the appropriate Social Planner’s problem.

The Social Planner’s problem is:

\[
\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \\
\text{s.t.} \quad c_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t
\]

Lagrange:

\[
L = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) - \sum_{t=0}^{\infty} \lambda_t \left[ c_t + k_{t+1} - F(k_t, h_t) - (1-\delta)k_t \right]
\]

First order necessary conditions:

First order conditions:

\[
[c_t]: \quad \beta^t u_1(c_t,1-h_t) - \lambda_t = 0 \\
[h_t]: \quad -\beta^t u_2(c_t,1-h_t) + \lambda_t F_2(k_t,h_t) = 0 \\
[k_{t+1}]: \quad -\lambda_t + \lambda_{t+1} \left[ F_1(k_{t+1},h_{t+1}) + 1-\delta \right] = 0
\]

Combining the conditions for \( c_t \) and \( h_t \), gives:

\[
\frac{\beta^t u_2(c_t,1-h_t)}{\beta^t u_1(c_t,1-h_t)} = \frac{\lambda_t F_2(k_t,h_t)}{\lambda_t} \\
(1) \quad \frac{u_2(c_t,1-h_t)}{u_1(c_t,1-h_t)} = F_2(k_t,h_t)
\]

Using the condition for consumption at time \( t \) and \( t+1 \):

\[
\frac{\beta^t u_1(c_t,1-h_t)}{\beta^{t+1} u_1(c_{t+1},1-h_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}}
\]

Combining with the condition for \( k_{t+1} \) gives the Euler equation:

\[
(2) \quad \frac{u_1(c_t,1-h_t)}{u_1(c_{t+1},1-h_{t+1})} = \beta [F_1(k_{t+1},h_{t+1}) + 1-\delta]
\]
The last equation is the feasibility constraint, which can be obtained from differentiating the Lagrange function with respect to $\lambda_t$, or simply by noting that the feasibility constraint of the social planner is the same as the feasibility in a competitive equilibrium setting.

$$c_t + k_{t+1} = F(k_t, h_t) + (1 - \delta)k_t$$

Thus, the competitive equilibrium allocation solves the Social Planner’s problem, and therefore it must be Pareto Optimal (recall that any Pareto Optimal allocation is a solution to some Social Planner’s problem).
2. (20 points). Suppose the Euler Equation from NGM with inelastic labor supply is:

\[ [EE]: \frac{u'(c_t)}{u'(c_{t+1})} = \beta R_{t+1} \]

where \( R_{t+1} = r_{t+1} + 1 - \delta \) is gross real interest rate. Let the utility from consumption be of the constant inter-temporal elasticity of substitution form:

\[ u(c_t) = \begin{cases} 
\frac{c_t^{1-\gamma} - 1}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\
\ln(c_t) & \gamma = 1 
\end{cases} \]

a. (10 points). Show that the Arrow-Pratt measure of relative risk aversion is \( RRA = \gamma \) and the inter-temporal elasticity of substitution is \( ES = 1/\gamma \).

**A-P measure of relative risk aversion**

\[ u'(c) = c^{-\gamma} \]

\[ u''(c) = -\gamma c^{-\gamma-1} \]

\[ RRA = -\frac{u''(c)}{u'(c)}c = -\frac{\gamma c^{-\gamma-1}}{c^{-\gamma}}c = \gamma \]

**Elasticity of substitution:**

\[ MRS = \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{c_t^{-\gamma}}{\beta c_{t+1}^{-\gamma}} = \frac{1}{\beta} \left( \frac{c_{t+1}}{c_t} \right)^{\gamma} \]

\[ \frac{1}{ES} = \frac{dMRS}{d(c_{t+1}/c_t)} \frac{c_{t+1}/c_t}{MRS} = \frac{1}{\beta} \gamma \left( \frac{c_{t+1}}{c_t} \right)^{\gamma-1} \frac{c_{t+1}/c_t}{\frac{1}{\beta} \left( \frac{c_{t+1}}{c_t} \right)^{\gamma}} = \gamma . \]

\[ \Rightarrow ES = \frac{1}{\gamma} \]
b. (10 points). Prove that the higher the inter-temporal elasticity of substitution is the higher is the response of consumption growth to changes in $R_{t+1}$, and provide economic intuition for it. Use the Euler equation and utility function given in the last section.

With the given utility function, we have

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_t^{-\gamma}}{c_{t+1}^{-\gamma}} = \left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$$

Thus, the Euler Equation is:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta R_{t+1}$$

Taking logs of both sides, gives:

$$\gamma \log\left(\frac{c_{t+1}}{c_t}\right) = \log \beta + \log\left(R_{t+1}\right)$$

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma} \cdot \log \beta + \frac{1}{\gamma} \cdot \log\left(R_{t+1}\right)$$

$$\log\left(1 + g_c\right) = ES \cdot \log \beta + ES \cdot \log\left(R_{t+1}\right)$$

where $g_c = \frac{c_{t+1} - c_t}{c_t}$ is the growth rate of consumption.

Note that the elasticity of $1 + g_c$ with respect to $R_{t+1}$ is given by the inter-temporal elasticity of substitution $ES$. Therefore the higher the inter-temporal elasticity of substitution is the higher is the response of consumption growth to changes in $R_{t+1}$

**Intuition.** Higher elasticity of substitution (lower risk aversion) means that the household is more willing to substitute current consumption for future consumption, when the relative price changes. The real gross interest rate $R_{t+1}$ is the relative price of current consumption in terms of future consumption.
3. (20 points). Consider the stochastic version of the Neoclassical Growth Model discussed in class. In particular, suppose that household’s period utility is 

\[ u(c_t, 1-h_t) \] 

and the lifetime expected utility is 

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \] 

The output is produced using technology 

\[ y_t = A_t k_t^{\theta} h_t^{1-\theta}, \] 

where 

\[ A_t = A_0 (1 + \gamma_A)^t e^{\xi_t} \] 

\[ z_t = \rho z_{t-1} + \epsilon_t, \epsilon_t \sim i.i.d. N(0, \sigma^2_{\epsilon}) \]. 

The law of motion of capital is standard: 

\[ k_{t+1} = (1-\delta)k_t + x_t, \] \[ k_0 > 0 \] is given.

a. (5 points). Write the social planner’s problem for this economy.

\[
\max \sum_{t=0}^{\infty} E_0 \beta^t u(c_t, 1-h_t) \\
\text{s.t.} \\
\begin{align*}
\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty} & \\
A_t k_{t-1}^{\theta} h_t^{1-\theta} + (1-\delta)k_t & = (1-\delta)k_t + x_t, \quad k_0 > 0 \text{ given} \\
A_t = A_0 (1 + \gamma_A)^t e^{\xi_t}, \quad z_t = \rho z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d. N(0, \sigma^2_{\epsilon}) 
\end{align*}
\]

b. (5 points). The reason why economists detrend the data is (circle the correct answer):

i. to force the data to match an AR(1) process.

ii. to separate the business cycle fluctuations from the long-run growth trend.

iii. to improve the calibration procedure.

iv. none of the above.
c. (10 points). Suppose that a calibrated version of the model in this question, was simulated and a comparison table of moments from the data and the model is reported below.

<table>
<thead>
<tr>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Y</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>I</td>
</tr>
<tr>
<td>H</td>
</tr>
<tr>
<td>w</td>
</tr>
</tbody>
</table>

Fill in the calculations in the empty spaces below.

i. The model accounts for ___80___% of the observed volatility in output.

ii. The model accounts for ___75___% of the observed volatility in consumption.

iii. The model accounts for ___150___% of the observed volatility in investment.

iv. The model accounts for ___50___% of the observed volatility in hours.

v. The model accounts for ___33.3___% of the observed volatility in wages.

vi. The data is consistent with consumption smoothing. True/false, circle the correct answer.

vii. The model is consistent with consumption smoothing. True/false, circle the correct answer.
4. (10 points). Inflation tax at time $t$ is defined as $IT_t = \frac{M_{t-1}}{P_{t-1}} - \frac{M_{t-1}}{P_t}$, and inflation rate is $\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}$.

a. (5 points). Prove that inflation tax can be written as $IT_t = \frac{M_{t-1}}{P_{t-1}} \left( \frac{\pi_t}{1 + \pi_t} \right)$, where $\frac{M_{t-1}}{P_{t-1}}$ is the tax base and $\frac{\pi_t}{1 + \pi_t}$ is the tax rate.

Starting from the definition of inflation tax, and plug $P_t = (1 + \pi_t)P_{t-1}$:

$IT_t = \frac{M_{t-1}}{P_{t-1}} - \frac{M_{t-1}}{P_t} = \frac{M_{t-1}}{P_{t-1}} \left( \frac{1 + \pi_t}{1 + \pi_t} \right) - \frac{M_{t-1}}{(1 + \pi_t)P_{t-1}}$

Rearranging in the required form:

$IT_t = \frac{M_{t-1}}{P_{t-1}} \left( 1 - \frac{1}{1 + \pi_t} \right) = \frac{M_{t-1}}{P_{t-1}} \left( \frac{1 + \pi_t - 1}{1 + \pi_t} \right) = \frac{M_{t-1}}{P_{t-1}} \left( \frac{\pi_t}{1 + \pi_t} \right)$

b. (5 points). Suppose that inflation rate is equal to 20%. What fraction of the purchasing power of money is lost due to inflation?

i. $\frac{1}{5}$

ii. $\frac{1}{6}$

iii. $\frac{2}{3}$

iv. $\frac{3}{10}$
5. (10 points). Suppose that the money supply grows at constant rate $\mu$, and the demand for real balances is

$$ \frac{M_t}{P_t} = \frac{1}{\sqrt{\pi_t}} $$

a. (5 points). Derive the amount of real revenues the government can collect as a function of $\mu$, assuming that inflation rate in the long run is equal to the growth rate of money: $\pi = \mu$.

$$ SE_t = \frac{M_t - M_{t-1}}{P_t} = \frac{M_t - M_{t-1}}{M_t} \cdot \frac{M_t}{P_t} $$

$$ = \frac{M_{t-1}(1 + \mu) - M_{t-1}}{M_{t-1}(1 + \mu)} \cdot \frac{1}{\sqrt{\pi}} $$

$$ = \frac{1 + \mu - 1}{(1 + \mu)} \cdot \frac{1}{\sqrt{\pi}} $$

$$ = \frac{\sqrt{\mu}}{1 + \mu} $$
b. (5 points). Find the rate of money growth which maximizes the seigniorage.

\[
\max SE = \frac{\sqrt{\mu}}{1 + \mu}
\]

First order condition:

\[
\frac{1}{2} \mu \left( \frac{1}{2} (1 + \mu) - \sqrt{\mu} \right) \frac{1}{(1 + \mu)^2} = 0
\]

Thus,

\[
\frac{1}{2} \mu \left( \frac{1}{2} (1 + \mu) \right) = \sqrt{\mu}
\]

\[
1 + \mu = 2 \mu
\]

\[
\mu = 1 = 100\%
\]
6. (10 points). Consider the Cash In Advance (CIA) model discussed in class. The household’s problem is:

\[
\max_{\{c_t, h_t, b_{t+1}, m_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)
\]

[B.C.]: \( p_t c_t + m_{t+1} + b_{t+1} = m_t + (1 + i_t)b_t + p_t h_t + \tau_t \)

[CIA]: \( p_t c_t = m_t \)

The solution is characterized by the following conditions (derived in class):

(1). \[
\frac{u_1(c_t, 1 - h_t)}{u_1(c_{t+1}, 1 - h_{t+1})} = \beta \frac{p_t (1 + i_t)}{p_{t+1}}
\]

(2). \[
\frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = \frac{p_t}{p_t (1 + i_t)}
\]

We argued in class that the above conditions are not Pareto Optimal due to the CIA constraint, unless \( i_t = 0 \), i.e. Friedman rule monetary policy. Now you are asked to show this formally. In particular, write the Social Planner’s problem, which is identical to the above consumer’s problem, but without the CIA constraint. Show that the solution the social planner’s problem is characterized by conditions (1SP) and (2SP), which are the same as (1) and (2) above, but with with \( i_t = 0 \) (Friedman rule).

Social planner’s problem:

\[
\max_{\{c_t, h_t, b_{t+1}, m_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)
\]

[B.C.]: \( p_t c_t + m_{t+1} + b_{t+1} = m_t + (1 + i_t)b_t + p_t h_t + \tau_t \)

Lagrange function:

\[
L = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) - \sum_{t=0}^{\infty} \lambda_t [p_t c_t + m_{t+1} + b_{t+1} - m_t - (1 + i_t)b_t - p_t h_t - \tau_t]
\]

F.O.C.s (for all \( t = 0,1,2,... \))

\[
\begin{align*}
[c_t]: & \quad \beta^t u_1(c_t, 1 - h_t) - \lambda_t p_t = 0 \\
[h_t]: & \quad - \beta^t u_2(c_t, 1 - h_t) + \lambda_t p_t = 0 \\
[b_{t+1}]: & \quad - \lambda_t + \lambda_{t+1}(1 + i_{t+1}) = 0 \\
[m_{t+1}]: & \quad - \lambda_t + \lambda_{t+1} = 0
\end{align*}
\]

The last two conditions imply the Friedman rule (\( i_t = 0 \)). From the conditions \([c_t]\) and \([c_{t+1}]\), we get the Euler Equation:

\[
\frac{\beta^t u_1(c_t, 1 - h_t)}{\beta^{t+1} u_1(c_{t+1}, 1 - h_{t+1})} = \frac{\lambda_t p_t}{\lambda_{t+1} p_{t+1}}
\]
Since $\lambda_t = \lambda_{t+1}$, the above becomes:

\[
(1SP). \quad \frac{u_1(c_t, l - h_t)}{u_1(c_{t+1}, l - h_{t+1})} = \beta \frac{p_t}{p_{t+1}}
\]

Combining the conditions $[c_t]$ and $[h_t]$, gives

\[
(2SP). \quad \frac{u_2(c_t, l - h_t)}{u_1(c_t, l - h_t)} = \frac{p_t}{p_t} = 1
\]
7. (20 points). Consider a representative investor, who lives forever, derives utility from consumption, and earns income from wages \( w_t \), and asset returns. His portfolio at time \( t \) consists of \( n \) assets, indexed \( i = 1,2,\ldots,n \), where \( x_{it} \) is the number of units of asset \( i \) in the portfolio. The price of asset \( i \) at time \( t \) is \( p_{it} \) and the dividend paid per unit of asset is \( d_{it} \) (both are in units of consumption good). Future returns on assets, and future income, are unknown at present time, and therefore the investor solves the maximization of expected lifetime utility problem:

\[
\max_{\{c_t, x_{it+1}\}_{t=1}^{T} \in \mathbb{N}; \sum_{t=0}^{\infty} \beta^t u(c_t)} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

s.t.

\[
c_t + \sum_{i=1}^{n} p_{it} x_{it+1} = w_t + \sum_{i=1}^{n} p_{it} x_{it} + \sum_{i=1}^{n} x_{it} d_{it} \quad \forall t = 0,1,\ldots
\]

a. (5 points). Derive the stochastic Euler equation for the optimal investment in asset \( i \).

Lagrange function:

\[
L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t \left[ c_t + \sum_{i=1}^{n} p_{it} x_{it+1} - w_t - \sum_{i=1}^{n} p_{it} x_{it} - \sum_{i=1}^{n} x_{it} d_{it} \right] \right\}
\]

F.O.C.s

\[
[c_t]: \beta^t u'(c_t) - \lambda_t = 0
\]

\[
[x_{it+1}]: - \lambda_t p_{it} + E_t \left[ \lambda_{t+1}(p_{it+1} + d_{it+1}) \right] = 0
\]

Substituting the first into the second, and rearranging, gives the Euler Equation:

\[
[EE]: u'(c_t) p_{it} = \beta E_t [u'(c_{t+1})(p_{it+1} + d_{it+1})]
\]
b. (5 points). Provide economic intuition for the Euler Equation derived in the last section.

Investing one extra unit in asset \( i \) entails giving up \( p_{it} \) units of current consumption \( c_t \). Thus, the left hand side represents the utility loss ("pain") from such investment. In the next period, the return on the investment is \( p_{it+1} + d_{it+1} \) (price of the asset + dividends), which is the gain in future consumption. Multiplication by marginal utility, translates the gain in consumption into utility gain. Therefore, the right hand side is the present value of expected future gain from investing one unit in asset \( i \).

c. (5 points). Derive the asset pricing formula from the Euler Equation. That is, show that the price of any asset (in this model) must be equal to the expected discounted future return on the asset (future price + dividend),

\[
p_{it} = E_t \left[ m_{t+1} (p_{it+1} + d_{it+1}) \right], \quad \text{where} \quad m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad \text{is the stochastic discount factor.}
\]

Dividing the Euler Equation by the marginal utility from consumption, and bringing \( \beta \) inside of the expectation operator, gives the asset pricing formula:

\[
[EE]: \quad u'(c_t) p_{it} = \beta E_t \left[ u'(c_{t+1}) (p_{it+1} + d_{it+1}) \right]
\]

\[
p_{it} = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} (p_{it+1} + d_{it+1}) \right]
\]

\[
[AP]: \quad p_{it} = E_t \left[ m_{t+1} (p_{it+1} + d_{it+1}) \right]
\]
d. (5 points). Let the gross return on asset \( i \) be 
\[ 1 + r_{it+1} = \frac{p_{it+1} + d_{it+1}}{p_{it}}. \] Using the asset pricing formula, show that the risk-free return is given by 
\[ 1 + r_{f,t+1} = \frac{1}{E_t(m_{t+1})}. \]

Dividing the Asset Pricing equation (AP) by the current price \( p_{it} \), gives the asset pricing formula in terms of asset returns:
\[
1 = E_t \left[ m_{t+1} \left( \frac{p_{it+1} + d_{it+1}}{p_{it}} \right) \right] = E_t \left[ m_{t+1} (1 + r_{it+1}) \right]
\]

Substituting the risk-free return and rearranging, gives:
\[
1 = E_t \left[ m_{t+1} (1 + r_{f,t+1}) \right] = E_t \left[ m_t (1 + r_{t+1}) \right]
\]
\[ \Rightarrow 1 + r_{f,t+1} = \frac{1}{E_t(m_{t+1})} \]