Final exam

Wednesday, May 22

1 hours, 30 minutes

Name: ___________________________________

Instructions

1. This is closed book, closed notes exam.

2. No calculators of any kind are allowed.

3. Show all the calculations.

4. If you need more space, use the back of the page.

5. Fully label all graphs.

6. Derive means that you need to show steps

7. Write means that you don’t need to show steps, just write the result.

Good Luck 😊
1. (15 points). Consider the Solow model, briefly described as follows. Output is produced according to $Y_t = F(K_t, L_t)$, where $F$ has all the properties of the Neoclassical production function. Capital evolves according to $K_{t+1} = K_t(1 - \delta) + I_t$, where $\delta$ is the depreciation rate and $I_t$ is aggregate investment. People save a fraction $s$ of their income. This fraction is exogenous. Thus, the total saving and total investment in this economy is $S_t = I_t = sY_t$. The population of workers grows at a constant rate of $n$, which is exogenous in this model. Thus, $L_{t+1} = (1 + n)L_t$.

   a. Let $k_t = \frac{K_t}{L_t}$ be capital per worker. Prove that output per worker can be expressed as a function of $k_t$ only, i.e. $f(k_t)$. 

   $\frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = F(k_t, 1)$ 

   The second step follows from the assumption that $F$ has Constant Returns to Scale (CRS). Then, we define $f(k_t) \equiv F(k_t, 1)$.

   b. The following graph plots certain flows for the Solow model. On the graph, indicate the steady state capital per worker ($k_{ss}$), output per worker ($y_{ss}$) and consumption per worker ($c_{ss}$). 

![Graph showing the Solow model flows with output, consumption, saving, and capital per worker axes.](image)
c. Suppose that the Chinese economy is described by the Solow model, and the graph in the previous section represents the calibrated Solow model to the Chinese economy. Based on this information, an economist from the Bank of China claims that the saving rate in China is “too high”. Explain briefly why the economist is correct (based on the Solow model of course). In your explanation, use the graph from the last section, which is reproduced here for your convenience.

The saving rate is “too high” because it is possible to increase the steady state consumption by lowering the saving rate. The above figure shows the highest possible steady state consumption per worker, $c_{GR}$, the golden rule consumption per worker, which is higher than the current steady state of the economy.
2. (15 points). Consider the Neoclassical Growth Model discussed in class. There is a single representative household and a single representative firm, that live forever. The household’s period utility function \( u(c_t, l_t) \), and the lifetime utility is \( U(\{c_t, l_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \), where \( 0 < \beta < 1 \) is the discount factor. The household has 1 unit of time, so the labor supply is \( h_t = 1 - l_t \). The household owns the capital stock \( k_t \), with the law of motion \( k_{t+1} = (1-\delta)k_t + x_t \), where \( x_t \) is investment and \( k_0 > 0 \) is given. The household receives a real wage \( w_t \) per unit of labor supplied to the firm, a rental rate \( r_t \) per unit of capital stock rented to the firm, and \( \pi_t \) - the profit (dividend) from the representative firm. There is a single representative firm that produces the output in this economy, with production function \( y_t = F(K_t, L_t) \), where \( K_t \) is capital and the \( L_t \) is labor. Assume that \( F(K_t, L_t) \) satisfies all the assumptions of a neoclassical production function. The economy is closed and there is no government, thus the feasibility constraint is: \( c_t + x_t = y_t \).

a. Write the household’s lifetime utility maximization problem.

\[
\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \\
\text{ s.t. } \quad c_t + k_{t+1} = w_t h_t + r_t k_t + (1-\delta)k_t + \pi_t
\]

Notice that the profit is zero since the production function has constant returns to scale. Also notice that we substituted the leisure from the time constraint: \( l_t = 1 - h_t \), and the choice variable now is \( h_t \) - the labor supplied (work time).
b. Write the Lagrange function corresponding to the household’s problem in the last section, and derive the optimality conditions for labor supplied and investment (Euler equation).

\[ L = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) - \sum_{t=0}^{\infty} \lambda_t \left[ c_t + k_{t+1} - w_t h_t - r_t k_t - (1 - \delta) k_t \right] \]

First order conditions:

\[ [c_t]: \quad \beta^t u_1(c_t, 1 - h_t) - \lambda_t = 0 \]

\[ [h_t]: \quad -\beta^t u_2(c_t, 1 - h_t) + \lambda_t w_t = 0 \]

\[ [k_{t+1}]: \quad -\lambda_t + \lambda_{t+1} [r_{t+1} + 1 - \delta] = 0 \]

Combining the conditions for \( c_t \) and \( h_t \), gives:

\[ \frac{\beta^t u_2(c_t, 1 - h_t)}{\beta^t u_1(c_t, 1 - h_t)} = \frac{\lambda_t}{\lambda_t} w_t \]

\[ \frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = w_t \]

Using the condition for consumption at time \( t \) and \( t+1 \):

\[ \frac{\beta^t u_1(c_t, 1 - h_t)}{\beta^{t+1} u_1(c_{t+1}, 1 - h_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}} \]

Combining with the condition for \( k_{t+1} \) gives the Euler equation:

\[ \frac{u_1(c_t, 1 - h_t)}{u_1(c_{t+1}, 1 - h_{t+1})} = \beta [r_{t+1} + 1 - \delta] \]

To summarize, the optimality conditions are:

1. \[ \frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = w_t \]

2. \[ \frac{u_1(c_t, 1 - h_t)}{u_1(c_{t+1}, 1 - h_{t+1})} = \beta [r_{t+1} + 1 - \delta] \]
c. Provide economic intuition for the optimal investment condition (Euler equation), after rewriting it in the form that shows the marginal “pain” and “gain” from investment.

Equation (2) in the last section can be written as:

\[ u_1(c_t, 1 - h_t) = \beta u_1(c_{t+1}, 1 - h_{t+1})[r_{t+1} + 1 - \delta] \]

The left hand side is the “pain” (decline in utility) as a result of investing extra unit of income in physical capital (and therefore giving up 1 unit of consumption) in period \( t \).

Recall that the marginal utility of consumption is the utility is the change in utility resulting from 1 unit change in consumption.

The right hand side is the utility gain from that investment. In period \( t + 1 \), the return on this investment, in units of consumption, is equal to \( r_{t+1} \) + the non-depreciated unit of capital originally created. To convert this return into utility we multiply by the marginal utility from consumption, and to convert to present value we multiply by the discount factor.

Thus, the optimal investment condition requires balancing the marginal pain and the marginal gain from investment. Any model of investment must have a condition similar to this one.
3. (15 points). Suppose that utility from certain amounts of money is given by $u(x)$, and utility from lottery can be represented as expected utility over the prizes $E[u(x)]$.
   a. Define a risk averse individual.

   An individual is risk averse if he/she prefers the expected prize of the lottery with certainty, over participating in the lottery:
   
   $$u[E(x)] > E[u(x)]$$

   b. Suppose that utility from certain amounts is CRRA:

   $$u(x) = \begin{cases} 
   \frac{x^{1-\gamma} - 1}{1-\gamma} & \gamma > 0, \; \gamma \neq 1 \\
   \ln(x) & \gamma = 1 
   \end{cases}$$

   Calculate the Arrow-Pratt coefficient of relative risk aversion (RRA) for this utility function.

   $$u'(x) = x^{-\gamma}$$
   $$u''(x) = -\gamma x^{-\gamma-1}$$

   $$RRA = \frac{u''(x)}{u'(x)} x = -\frac{\gamma x^{-\gamma-1}}{x^{-\gamma}} x = \gamma$$
c. Suppose that consumer’s lifetime utility in the Neoclassical Growth Model is \( U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t) \), where \( u(\cdot) \) is given in the previous section.

Find the intertemporal elasticity of substitution, \( IES_{c_t,c_{t+1}} \) (i.e. elasticity of substitution between \( c_t \) and \( c_{t+1} \)).

\[
MRS_{c_t,c_{t+1}} = \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{c_t^{-\gamma}}{\beta c_{t+1}^{-\gamma}} = \frac{1}{\beta} \left( \frac{c_{t+1}}{c_t} \right)^{\gamma}
\]

\[
\frac{1}{IES_{c_t,c_{t+1}}} = \frac{dMRS_{c_t,c_{t+1}}}{d\left(\frac{c_{t+1}}{c_t}\right)} \cdot \frac{c_{t+1}}{c_t} = \frac{1}{\beta} \gamma \left( \frac{c_{t+1}}{c_t} \right)^{\gamma-1} \cdot \frac{c_{t+1}}{c_t} = \gamma
\]

\[\Rightarrow IES_{c_t,c_{t+1}} = \frac{1}{\gamma}\]
4. (15 points). Suppose that Stephen owns some wealth \( w \), and he faces a risk with probability \( \pi \) of losing an amount of \( d \in (0, w) \) (damage). Stephen can buy insurance, with premium \( p \) per unit of coverage. Stephen’s problem is therefore to choose the amount of coverage to buy: \( 0 \leq q \leq d \). This means that he pays premium \( pq \) and in case of damage the insurance company will pay him \( q \). His utility from wealth is given by \( u(\cdot) \), which is strictly increasing and strictly concave function.

a. Write Stephen’s problem of choosing optimal coverage, and derive the first order necessary condition for interior optimum. Denote the optimal coverage by \( q^* \).

\[
\max_{0 \leq q \leq d} E[u] = \pi u(w - d - pq + q) + (1 - \pi)u(w - pq) \\
\frac{dE[u]}{dq} = \pi u'(w - d - pq^* + q^*)(1 - \pi) - (1 - \pi)u'(w - pq^*)p = 0
\]

Where \( q^* \) denotes the optimal \( q \).

b. Prove that if Stephen is risk averse, then the above first order condition indeed determines the optimal coverage.

The second order sufficient condition for global maximum is that the objective function \( E[u(q)] \) be strictly concave.

\[
\frac{d^2 E[u(q)]}{dq^2} = \pi u''(w - d - pq + q)(1 - \pi)^2 + (1 - \pi)u''(w - pq)p^2 < 0
\]

This condition is satisfied for all \( q \) when \( u'' < 0 \), i.e. Stephen is risk averse.

c. Suppose that Stephen is risk averse. Prove that if the insurance is actuarially fair, then Stephen will buy full coverage.

Actuarially fair insurance means that \( p = \pi \), i.e. the premium per unit of coverage is equal to the probability of damage. Using the first order condition and \( p = \pi \), we get:

\[
\pi u'(w - d - \pi q^* + q^*)(1 - \pi) = (1 - \pi)u'(w - \pi q^*)
\]
\[
u'(w - d - \pi q^* + q^*) = u'(w - \pi q^*)
\]
\[
w - d - \pi q^* + q^* = w - \pi q^*
\]
\[-d + q^* = 0
\]
\[q^* = d\]
5. (15 points). Consider the stochastic version of the Neoclassical Growth Model discussed in class. In particular, suppose that household’s period utility is 
\[ u(c_t, 1-h_t) \] and the lifetime expected utility is \[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \]. The output is produced using technology \[ y_t = A_t k_t^\theta h_t^{1-\theta} \], where \[ A_t = A_0 (1 + \gamma_A) e^{\xi_t} \] \[ z_t = \rho z_{t-1} + \epsilon_t, \epsilon_t \sim i.i.d. N(0, \sigma^2_\epsilon) \]. The law of motion of capital is standard: \[ k_{t+1} = (1-\delta)k_t + x_t, \] and \[ k_0 > 0 \] is given.

a. Write the social planner’s problem for this economy.

\[ \max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \]

s.t.

\[ c_t + k_{t+1} = A_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t, \quad k_0 > 0 \text{ given} \]

\[ A_t = A_0 (1 + \gamma_A) e^{\xi_t}, \quad z_t = \rho z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d. N(0, \sigma^2_\epsilon) \]
b. (5 points). Write the necessary conditions for competitive equilibrium in this model economy (i.e., 3 equations for every time period, that can be solved for equilibrium sequences of \(\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}\)). You do not need to derive anything, but you can if you want to. Your derivations will not be graded.

**Necessary conditions for equilibrium**

(1) \(\frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = (1-\theta)A_t k_t^{\theta} h_t^{-\theta}\)

(2) \(u_1(c_t, 1-h_t) = E_t \beta u_1(c_{t+1}, 1-h_{t+1})[\theta A_{t+1} k_{t+1}^{\theta} h_{t+1}^{1-\theta} + 1-\delta]\)

(3) \(c_t + k_{t+1} = A_t k_t^{\theta} h_t^{1-\theta} + (1-\delta)k_t\)

The above 3 conditions must hold for all time \(t = 0,1,2,...\)

Equation (2) is the stochastic Euler equation. Notice that equation (2) has \(E_t\), i.e. conditional expectation given the information available at time \(t\).
c. Consider the AR(1) process of the productivity process: $z_t = \rho z_{t-1} + \epsilon_t$, $\epsilon_t \sim i.i.d. N(0, \sigma^2)$, and $|\rho| < 1$. Prove that the $k$-step ahead prediction is $E(z_{t+k} | z_t) = \rho^k z_t$. In your proof, you are allowed to use the result we derived in class, that if $|\rho| < 1$, then $z_t = \epsilon_t + \rho \epsilon_{t-1} + \rho^2 \epsilon_{t-2} + ...$, i.e. that $z_t$ can be represented as a weighted sum of all the shocks up to and including time $t$.

Starting with the 1-step ahead prediction:

$$E(z_{t+1} \mid z_t) = E(\rho z_t + \epsilon_{t+1} \mid z_t) = \rho E(z_t \mid z_t) + E(\epsilon_{t+1} \mid z_t) = \rho z_t$$

The term $E(\epsilon_{t+1} \mid z_t) = E(\epsilon_{t+1}) = 0$ because $z_t$ depends on the $\epsilon$-s up to time $t$, so $\epsilon_{t+1}$ is independent of $z_t$. Thus, the conditional expectation is the same as unconditional, and it is given that $E(\epsilon_t) = 0 \ \forall t$.

Similarly, the 2-step ahead prediction:

$$E(z_{t+2} \mid z_t) = E(\rho z_{t+1} + \epsilon_{t+2} \mid z_t) = \rho E(z_{t+1} \mid z_t) + E(\epsilon_{t+2} \mid z_t) = \rho^2 z_t$$

Once again, the term $E(\epsilon_{t+2} \mid z_t) = E(\epsilon_{t+2}) = 0$ as explained earlier.

Repeating the above steps $k$ times, gives the desired result:

$$E(z_{t+k} \mid z_t) = \rho^k z_t$$
6. (25 points). Inflation tax at time \( t \) is defined as \( IT_t = \frac{M_{t-1}}{P_{t-1}} - \frac{M_{t-1}}{P_t} \), and inflation rate is \( \pi_t = \frac{P_t - P_{t-1}}{P_{t-1}} \).

a. Prove that \( IT_t = \frac{M_{t-1}}{P_{t-1}} \left( \frac{\pi_t}{1 + \pi_t} \right) \), where \( \frac{M_{t-1}}{P_{t-1}} \) is the tax base and \( \frac{\pi_t}{1 + \pi_t} \) is the tax rate.

\[
IT_t = \frac{M_{t-1}}{P_{t-1}} \left( 1 - \frac{P_{t-1}}{P_t} \right)
\]

Notice that \( 1 + \pi_t = \frac{P_t}{P_{t-1}} \). Thus, the inflation tax is:

\[
IT_t = \frac{M_{t-1}}{P_{t-1}} \left( 1 - \frac{1}{1 + \pi_t} \right) = \frac{M_{t-1}}{P_{t-1}} \left( \frac{1 + \pi_t - 1}{1 + \pi_t} \right) \]

\[
= \frac{M_{t-1}}{P_{t-1}} \left( \frac{\pi_t}{1 + \pi_t} \right)
\]

b. Calculate the inflation tax rate for the following inflation rates, i.e., complete the next table:

<table>
<thead>
<tr>
<th>( \pi_t )</th>
<th>50%</th>
<th>100%</th>
<th>200%</th>
<th>900%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation tax rate</td>
<td>( \frac{50}{150} = 33\frac{1}{3}% )</td>
<td>( \frac{100}{200} = 50% )</td>
<td>( \frac{200}{300} = 66\frac{2}{3}% )</td>
<td>( \frac{900}{1000} = 90% )</td>
</tr>
</tbody>
</table>
c. Suppose that the money supply grows at constant rate $\mu$, and the demand for real balances is

\[
\frac{M_t}{P_t} = \frac{a}{1 + \pi_t}, \quad a > 0
\]

Derive the amount of real revenues the government can collect as a function of $\mu$ (seigniorage), assuming that inflation rate in the long run is equal to the growth rate of money: $\pi = \mu$. Hint: start with

\[
SE_t = \frac{M_t - M_{t-1}}{P_t} = \frac{M_t - M_{t-1}}{M_t} \cdot \frac{M_t}{P_t},
\]

and express the seigniorage as a function of $\mu$.

\[
SE_t = \frac{M_t - M_{t-1}}{P_t} = \frac{M_t - M_{t-1}}{M_t} \cdot \frac{M_t}{P_t}
\]

\[
= \frac{M_{t-1}(1 + \mu) - M_{t-1}}{M_{t-1}(1 + \mu)} \left( \frac{a}{1 + \pi} \right)
\]

\[
= \frac{1 + \mu - 1}{(1 + \mu)} \left( \frac{a}{1 + \mu} \right)
\]

\[
= \frac{a \mu}{(1 + \mu)^2}
\]
d. Suppose that long-run seigniorage is \( SE = \frac{a \mu}{(1 + \mu)^2}, \ a > 0 \). Prove that in order to maximize the seigniorage, the inflation rate should be 100%.

\[
\max_{\mu} SE = \frac{a \mu}{(1 + \mu)^2}
\]

First order condition:

\[
a \left[ \frac{(1 + \mu)^2 - 2\mu(1 + \mu)}{(1 + \mu)^4} \right] = 0
\]

\[
(1 + \mu)^2 = 2\mu(1 + \mu)
\]

\[
1 + \mu = 2\mu
\]

\[
\mu = 1 = 100\%
\]

e. Suppose that long-run seigniorage is \( SE = \frac{a \mu}{(1 + \mu)^2}, \ a > 0 \). Prove that \( \lim_{\mu \to \infty} SE = 0 \).

- Using L'Hôpital's rule:

\[
\lim_{\mu \to \infty} \frac{a \mu}{(1 + \mu)^2} = \lim_{\mu \to \infty} \frac{a}{2(1 + \mu)} = 0
\]

- Without L'Hôpital's rule:

\[
\lim_{\mu \to \infty} \frac{a \mu}{1 + 2\mu + \mu^2} = \lim_{\mu \to \infty} \frac{a}{\mu + 2 + \mu} = \lim_{\mu \to \infty} \frac{a}{2 + \mu} = 0
\]
7. (10 points). Consider the Cash In Advance (CIA) model discussed in class. The household's problem is:

\[ \max_{\{c_t, h_t, m_t, \tau_t \}} \sum_{t=0}^{\infty} \beta^t (c_t, 1 - h_t) \]

[B.C.]: \( p_t c_t + m_{t+1} + b_{t+1} = m_t + (1 + i_t) b_t + p_t h_t + \tau_t \)

[CIA]: \( p_t c_t = m_t \)

a. What is the purpose of the CIA constraint in this model?

Without the CIA constraint, people would not like to hold money. CIA constraint is one of the ways economists force money into the model. Another popular alternative to CIA is **money in the utility function**.

b. (5 points). Consider a steady state with constant growth rate of money supply \( \mu \). Prove that inflation rate is equal to the growth rate of money: \( \pi = \mu \).

Using the CIA constraint:

\[
\frac{p_{t+1} c_{t+1}}{p_t c_t} = \frac{m_{t+1}}{m_t} = 1 + \mu
\]

\[
\frac{p_{t+1} c}{p_t c} = 1 + \mu
\]

\[
1 + \pi = 1 + \mu
\]

\[
\pi = \mu
\]