Consumption Based Asset Pricing

1 Introduction

Traditional finance studies financial markets and asset prices in isolation from other markets for goods and services. Dynamic General Equilibrium theory used in macro economics (Stochastic Neoclassical Growth Model) however, is a framework that allows studying all markets and all trades in goods, services, and financial assets jointly. These notes provide a short introduction into the unified theory of macroeconomics and finance - macrofinance.

2 Model

The model we use is similar to the stochastic NGM, with inelastic labor supply. The representative household earns income from wages, $w_t$, and from $i = 1, \ldots, n$ assets. Asset holding at time $t$ is $x_{it}$, where $i = 1, 2, \ldots, n$ are indices of assets. Asset prices are $p_{it}$, and dividends $d_{it}$. The household’s problem is:

$$\max_{\{c_t, x_{it+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c_t + \sum_{i=1}^{n} p_{it} x_{it+1} = w_t + \sum_{i=1}^{n} p_{it} x_{it} + \sum_{i=1}^{n} d_{it} x_{it} \quad \forall t$$

The expectation operator is there because we allow future income, prices and dividends to be uncertain. The Lagrange function is:

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t \left[ c_t + \sum_{i=1}^{n} p_{it} x_{it+1} - w_t - \sum_{i=1}^{n} p_{it} x_{it} - \sum_{i=1}^{n} d_{it} x_{it} \right] \right\}$$

F.O.C.

$$\begin{bmatrix} [c_t] : \beta^t u'(c_t) - \lambda_t = 0 & \forall t = 1, 2, \ldots \\ [x_{it+1}] : -\lambda_t p_{it} + E_t \lambda_{t+1} (p_{it+1} + d_{it+1}) = 0 & \forall i = 1, \ldots, n; \quad \forall t = 1, 2, \ldots \end{bmatrix}$$

Combining the conditions for $c_t, c_{t+1}$ and $x_{it+1}$, gives the Euler Equation:

$$-\beta^t u'(c_t) p_{it} + E_t \left[ \beta^{t+1} u'(c_{t+1}) (p_{it+1} + d_{it+1}) \right] = 0$$

or

$$u'(c_t) p_{it} = \beta E_t [u'(c_{t+1}) (p_{it+1} + d_{it+1})]$$

The economic intuition behind the Euler Equation should be familiar, as it is very similar to the Euler Equation in the Stochastic Neoclassical Growth Model with one asset - physical
capital. Investing one extra unit in asset $i$ entails giving up $p_{it}$ units of current consumption $c_t$. Thus, the left hand side represents the utility loss ("pain") from such investment. In the next period, the return on the investment is $p_{it+1} + d_{it+1}$ (price of the asset + dividends), which is the gain in future consumption. Multiplication by marginal utility, translates the gain in consumption into utility gain. Therefore, the right hand side is the present value of expected future gain from investing one unit in asset $i$.

Rearranging, gives the consumption based asset pricing equation:

$$p_{it} = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} (p_{it+1} + d_{it+1}) \right]$$

(1)

The term

$$m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

is called the stochastic discount factor, and it is the marginal rate of substitution between $c_{t+1}$ and $c_t$.

The asset pricing formula (1) can be rewritten in terms of (gross) return on asset $i$:

$$1 + r_{it+1} = R_{it+1} = \frac{p_{it+1} + d_{it+1}}{p_{it}}$$

Plugging $R_{it+1}$ into (1), gives:

$$1 = E_t (m_{t+1} R_{it+1})$$

(2)

where $R_{it+1}$ is the gross return (such as 1.05) on asset $i$ and $r_{it}$ is the net return (such as 0.05 or 5%) on asset $i$. Equation (2) is a very general asset pricing formula. Most theories of financial asset pricing can be expressed in terms of this formula.

2.1 Pricing risk-free asset

Suppose that there exists an asset with net return $r_{it+1}^f$ (real interest rate), which is guaranteed with 100% certainty. Using the asset pricing formula (2) gives:

$$1 = E_t \left[ m_{t+1} \left( 1 + r_{it+1}^f \right) \right] = E_t \left( m_{t+1} \right) \left( 1 + r_{t+1}^f \right)$$

$$1 + r_{t+1}^f = \frac{1}{E_t (m_{t+1})} = \frac{1}{\beta E_t \left( \frac{u'(c_{t+1})}{u'(c_t)} \right)}$$

(3)

We have the following results:

1. Higher $\beta$ means that people are more patient (put more value on future consumption), and they are willing to accept lower interest rate when they trade current for future consumption.

2. Real interest rates are high when consumption growth is high. To see this, recall that marginal utility is diminishing, and write

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{u'(c_t (1 + g_{ct+1}))}{u'(c_t)}$$
The higher is the growth of consumption, $g_{ct+1} = \frac{c_{t+1} - c_t}{c_t}$, the lower is the above ratio, leading to higher $r_{t+1}^f$ in equation (3). Intuitively, with higher interest rates, current consumption becomes more expensive relative to future consumption, and consumers want to lower consumption today and invest more (consume more in the future).

3. Using first order (linear) Taylor expansion of $u'(c_{t+1})$ around $c_t$ gives

$$u'(c_{t+1}) \approx u'(c_t) + u''(c_t)(c_{t+1} - c_t)$$

Plugging this into equation (3)

$$1 + r_{t+1}^f \approx \frac{1}{\beta E_t \left[ \frac{u'(c_t) + u''(c_t)(c_{t+1} - c_t)}{u'(c_t)} \right]} = \frac{1}{\beta E_t \left[ \frac{1 + u''(c_t)(c_{t+1} - c_t)}{w'(c_t)} - c_t \right]}$$

$$1 + r_{t+1}^f \approx \frac{1}{\beta E_t \left[ 1 - RRA \cdot g_{ct+1} \right]} = \frac{1}{\beta \left[ 1 - RRA \cdot E_t (g_{ct+1}) \right]}$$

(4)

where $RRA$ is the Arrow-Pratt coefficient of relative risk aversion. Once again, we see that faster expected growth rate of consumption is associated with higher interest rates. However, the real interest rates are more sensitive to consumption growth when risk aversion, $RRA$, is higher. Intuitively, with higher risk aversion, consumers want smoother consumption path, and it takes larger interest rate to compensate them for a given consumption change. In a special case of risk neutrality, $RRA = 0$, we have

$$1 + r_{t+1}^f \approx \frac{1}{\beta} = 1 + \rho$$

Thus, real interest rate is equal to the utility discount factor $\rho$, and do not depend on consumption growth.

**Example 1** Suppose that $u(c) = \ln(c)$, the discount factor is $\beta = 0.97$, and expected future growth rate of consumption is $E_t (g_{ct+1}) = 2\%$. Using the approximate risk-free asset pricing formula (4), what should be the risk-free real interest rate?

**Solution 2** Recall that the relative risk aversion for logarithmic preferences is 1. Thus,

$$1 + r_{t+1}^f \approx \frac{1}{0.97 \left[ 1 - 1 \cdot 2\% \right]} = \frac{1}{0.97 \cdot 0.98} \approx 1.052$$

$$r_{t+1}^f \approx 5.2\%$$

Without risk aversion, we would have $RRA = 0$, and

$$r_{t+1}^f \approx 3.1\%$$
2.2 Pricing risky assets

Recall that from the definition of covariance between two random variable $X$ and $Y$, it follows:

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Applying this to asset pricing formula (2) and manipulating:

$$1 = E_t[m_{t+1}(1 + r_{it+1})]$$

$$1 = E_t(m_{t+1})E_t(1 + r_{it+1}) + \text{Cov}[m_{t+1}, (1 + r_{it+1})]$$

$$\frac{1}{E_t(m_{t+1})} = E_t(1 + r_{it+1}) + \frac{\text{Cov}[m_{t+1}, (1 + r_{it+1})]}{E_t(m_{t+1})}$$

$$1 + r^f_{t+1} = E_t(1 + r_{it+1}) + \frac{\text{Cov}[m_{t+1}, (1 + r_{it+1})]}{E_t(m_{t+1})}$$

$$E_t(r_{it+1}) = r^f_{t+1} - \frac{\text{Cov}\left[\beta\frac{u(c_{t+1})}{u'(c_t)}, (1 + r_{it+1})\right]}{E_t\left[\beta\frac{u(c_{t+1})}{u'(c_t)}\right]}$$

$$E_t(r_{it+1}) = r^f_{t+1} - \frac{\text{Cov}[u'(c_{t+1}), (1 + r_{it+1})]}{E_t(u'(c_{t+1}))}$$

The first term in (5) is the risk free return, and the second term in (5) is risk adjustment or risk premium. Thus, if an asset return is uncorrelated with consumption, its expected return is equal to the risk free return, and there is no premium. Since $u'(c)$ is diminishing, asset returns that are positively correlated with consumption are negatively correlated with marginal utility. Thus, assets that are positively correlated with consumption, and therefore $\text{Cov}(u'(c_{t+1})(1 + r_{it+1})) < 0$, must promise higher expected return (because these assets make consumption more volatile, or increase risk). On the other hand, assets that are negatively correlated with consumption, and therefore $\text{Cov}(u'(c_{t+1})(1 + r_{it+1})) > 0$, can offer expected returns that are lower than the risk free return (because they help smooth consumption, or reduce risk - serve as insurance).

2.3 Bubbles

In this section we illustrate the possibility of speculative price bubbles. Using the asset pricing equation (1) and substituting $p_{it+1}$ into the formula for $p_{it}$, gives
\[ p_{it} = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} p_{it+1} + \beta \frac{u'(c_{t+1})}{u'(c_t)} d_{it+1} \right] \]

\[ = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} E_{t+1} \left( \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} p_{it+2} + \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} d_{it+2} \right) + \beta \frac{u'(c_{t+1})}{u'(c_t)} d_{it+1} \right] \]

\[ = E_t \left[ \beta^2 \frac{u'(c_{t+2})}{u'(c_t)} p_{it+2} + \beta^2 \frac{u'(c_{t+2})}{u'(c_t)} d_{it+2} + \beta \frac{u'(c_{t+1})}{u'(c_t)} d_{it+1} \right] \]

The last step uses the law of iterated expectations \( E_t \left[ E_{t+1} (X) \right] = E_t (X) \). If we keep substituting again for \( p_{it+2}, p_{it+3}, \ldots \) from the asset pricing equation (1), we obtain:

\[ p_{it} = \lim_{s \to \infty} E_t \left[ \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(c_s)}{u'(c_t)} d_{is} \right] + E_t \left[ \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(c_s)}{u'(c_t)} d_{is} \right] \]

The second term, \( f_t \), is the expected present discounted value of future dividends, where the discount factors are intertemporal marginal rates of substitution. The second term therefore represents the fundamental part of the asset price \( p_{it} \). The first term, \( b_t \), is usually assumed to be zero, and this assumption is equivalent to absence of speculative price bubble. However, in reality, there are cases where asset prices do not appear to be valued according to their fundamentals alone. Examples include the dot-com bubble in the 90s and the housing bubble of 2000-2006. A bubble in asset \( i \) exists when the first term \( b_t \neq 0 \). While some researchers conclude that the presence of bubbles is evidence of irrational behavior on the part of investors, other researchers have developed theories in which bubbles are consistent with perfectly rational behavior.

**References**