Problem set 5
Consumption-Saving Decision and Ricardian Equivalence

1. (10 points). Saving and Investment equation.
   a. Derive the saving and investment equation.

The formula for GDP using the expenditure approach is:
\[ GDP = C + I + G + NX \]  

- Defining disposable income: \( YD = GDP + TR - T \), where \( TR \) is transfer payments and \( T \) is total taxes net of subsidies.
- Defining private saving: \( S_p = YD - C \), that is the disposable income that is not spent on consumption.
- Defining government saving: \( S_G = T - TR - G \), that is the government income from taxes that is not spent on consumption and transfer payments.

Add the transfer payments and subtract the total taxes from (1):
\[ GDP + TR - T = C + I + G + TR - T + NX \]

Using these definitions, the above becomes:
\[ YD = C + I - S_G + NX \]
\[ YD - C - S_G = I + NX \]

**The Saving and Investment Equation:**

\[ S_p + S_G = I + NX \]

b. Suppose that in some economy the private saving is 15, the domestic investment is 20, and the government runs a deficit of 3. What must be the current account deficit in that country?

Using the saving and investment equation:
\[ S_p + S_G = I + NX \]
\[ 15 - 3 + 20 = ? \]

Thus, the balance on current account is \( NX = -8 \), and the current account deficit is therefore 8.

2. (35 points). In this question you need to use Excel and data for HW5 posted on the course web page. The questions that we address here are how big is the government debt, and what is the debt burden, i.e. how much interest does government pay because of the debt.
   a. Plot the graph of the government debt as a fraction of GDP for all the years for which the data is available.
b. Why does it make sense to look at the debt as a fraction of GDP, and not at the debt itself?

The debt relative to GDP tells us how big the debt is relative to the total income in the economy, and how difficult it is for that economy to pay its debt. This is the only sensible way to compare debt across time and between countries. Suppose that U.S. has 10 trillion dollars debt, and Greece has 1 trillion dollars debt, but the U.S. economy is 20 times larger than the economy of Greece. In this case, although U.S. has larger debt in dollar amount, the debt to GDP ratio in Greece is twice as large as the U.S. The following graph demonstrates this point.
c. What was the debt to GDP ratio during the last year of the data?

The debt in 2015 was about 73.7% of the GDP.

d. In what year was the size of the debt the largest, and in what year was the debt burden (debt/GDP ratio) the largest?

The largest debt was in 2015, of about 13116.692 billion (≈ 13.1 trillion)
The largest debt burden was in 1946, of about 106% of GDP.

e. Plot the graph of the interest payments as a fraction of government income (cost of debt) for all the years for which the data is available.

![Graph of Interest Payments](image)

f. Why does it make sense to look at the government interest payments as a percentage of its income?

What matters is the interest payments relative to the government’s income. The dollar value of any of the government expenditures is totally uninformative. If we want to learn anything about government spending, we must look at that spending as a fraction of government income. For example, spending on national defense, education, interest payments, all need to be presented as a fraction of government income.

g. In what year was the interest payment the largest, and in what year was the cost of debt (interest as a fraction of government receipts) the largest?

The largest interest payment was in 2008, of about 252.757 billion.
The largest interest as a fraction of government receipts was in 1991, of about 18.43% of GDP.
3. (40 points). Consider the two-period model of consumption and saving.

a. Write the consumer’s problem of utility maximization subject to the budget constraints in two periods.

\[
\begin{align*}
\text{max } U(c_1, c_2) \\
\text{s.t.} \\
BC_1: & \quad c_1 + s = y_1 - t_1 \\
BC_2: & \quad c_2 = y_2 - t_2 + (1 + r)s
\end{align*}
\]

b. Derive the lifetime budget constraint from the budget constraints in each period. Show your derivations.

Substitute \( s \) from the second period budget constraint into the first period’s budget constraint. It is easy to do when you divide both sides of \( BC_2 \) by \( 1 + r \) to get

\[
BC_2 : \quad \frac{c_2}{1 + r} = \frac{y_2}{1 + r} - \frac{t_2}{1 + r} + s
\]

Now add the two budget constraints and get the **lifetime budget constraint**:

\[
\frac{c_1 + c_2}{1 + r} = y_1 - t_1 + \frac{y_2 - t_2}{1 + r} \quad \text{PV of lifetime consumption} \quad \text{we = lifetime wealth}
\]

c. Give the economic interpretation of the left hand side and the right hand side of the lifetime budget constraint.

The left hand side is the present value of lifetime consumption, and the right hand side is the present value of lifetime net-of-taxes income, which we call the lifetime wealth (\( we \)).

d. Draw a fully labeled graph of the lifetime budget constraint with a tangent indifference curve indicating the optimal choice for a lender, and label the saving as well. In order to receive all the points your graph should be clear and the lines are carefully drawn with a ruler.
The next figure shows the optimal choice for a consumer who is a lender. The optimal choice (optimal consumption bundle) is at point A.

e. Draw a fully labeled graph of the lifetime budget constraint with a tangent indifference curve indicating the optimal choice for a borrower, and label the saving as well. In order to receive all the points your graph should be clear and the lines are carefully drawn with a ruler.
The next figure shows the optimal choice for a consumer who is a borrower. The optimal choice (optimal consumption bundle) is at point A.

![Diagram showing lifetime budget constraint and optimal choice]

f. An increase in current income \( y_1 \) will increase the current consumption \( c_1 \) by the same amount (i.e. \( \Delta c_1 = \Delta y_1 \)). True/False, circle the correct answer, and provide a proof.

We see from the lifetime budget constraint that an increase in \( y_1 \) will shift the budget constraint to the right. Given that both goods (current consumption and future consumption) are normal, the consumer will increase the consumption in both periods. In order to increase the consumption in the second period the consumer must increase his saving. Thus, an increase in the current income will increase the current consumption by less than the change in the current income. We call this result consumption smoothing.

To summarize: \( y_1 \uparrow \Rightarrow c_1^* \uparrow, s^* \uparrow, \Delta c_1 < \Delta y_1 \)

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g. Consider the government budget constraint and suppose that the real interest rate is 6%. If the government gives a tax cut of 30 in the first period (i.e. \( \Delta T_1 = -30 \)), find the necessary change in the second period’s taxes (\( \Delta T_2 = ? \)) that would keep the present value of taxes unchanged.
If the present value of government spending remains unchanged, then changes in the taxes do not affect the households’ optimal consumption choice \((c^*_1, c^*_2)\).

4. (10 points). Suppose that instead of a lump-sum taxes, the taxes are proportional to income \((0 < t_1, t_2 < 1)\), so that the budget constraints are now

\[
BC_1 : \quad c_1 + s = y_1 (1 - t_1) \\
BC_2 : \quad c_2 = y_2 (1 - t_2) + (1 + r)s
\]

Prove that the Ricardian Equivalence theorem still holds.

The consumer’s lifetime BC is:

\[
c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r} - \left[ t_1 y_1 + \frac{t_2 y_2}{1 + r} \right].
\]

From the government lifetime BC we see that

\[
G_1 + \frac{G_2}{1 + r} = N \left( t_1 y_1 + \frac{t_2 y_2}{1 + r} \right),
\]

thus, the lifetime tax liability of the consumers is still fixed when the present value of government spending is fixed.

5. (25 points). Consider the two-period model of consumption and saving. Suppose that the consumer’s utility is

\[
U(c_1, c_2) = \ln(c_1) + \beta \ln(c_2).
\]

a. Write the consumer’s problem

\[
\begin{align*}
\max_{c_1, c_2, s} & \quad \ln(c_1) + \beta \ln(c_2) \\
\text{s.t.} & \quad BC_1 : \quad c_1 + s = y_1 - t_1 \\
& \quad BC_2 : \quad c_2 = y_2 - t_2 + (1 + r)s
\end{align*}
\]

b. Write the consumer’s demand for consumption in both periods and his supply of saving.

Writing the consumer’s problem with the lifetime budget constraint helps for this section.
Now we can see that since the preferences are of the Cobb-Douglas form, the consumer will spend a fixed fraction of his lifetime income on $c_1$ and $c_2$:

$$c_1^* = \left( \frac{1}{1+\beta} \right) \left( y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right)$$

$$c_2^* = \left( \frac{\beta}{1+\beta} \right) \left( y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right) (1+r) = \left( \frac{\beta}{1+\beta} \right) \left( (y_1 - t_1)(1+r) + y_2 - t_2 \right)$$

The saving, from the first budget constraint

$$s^* = y_1 - t_1 - \left( \frac{1}{1+\beta} \right) \left( y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right)$$

\[ \text{c. Prove that saving for this consumer is increasing in real interest rate.} \]

Notice that $c_1^*$ is decreasing in $r$ (see section b), and $s^* = y_1 - t_1 - c_1^*$. Thus, $s^*$ is increasing in $r$.

Another way to show that $s^*$ is increasing in $r$ is to take the derivative

$$\frac{\partial s^*}{\partial r} = \left( \frac{1}{1+\beta} \right) \frac{y_2 - t_2}{(1+r)^2} > 0$$

(Obviously, the taxes are no greater than income, so the term $y_2 - t_2 > 0$, and the above derivative is always positive).

d. Suppose that income in the first period increases by $100. Find the change in the first period’s consumption.

$$c_1^* = \left( \frac{1}{1+\beta} \right) \left( y_1 + 100 - t_1 + \frac{y_2 - t_2}{1+r} \right) = \left( \frac{1}{1+\beta} \right) \left( y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right) + \left( \frac{1}{1+\beta} \right) \cdot 100$$

Thus, the change in $c_1^*$ is

$$\Delta c_1^* = \left( \frac{1}{1+\beta} \right) \cdot 100$$

e. How does the change in the change in $c_1$ in the last section depend on the parameter $\beta$? Provide economic intuition for this result.

We can see that higher $\beta$ implies smaller change in current consumption. Recall that $\beta$ represents the weight on future utility, and higher $\beta$ means that second period consumption becomes more important. Therefore, with higher $\beta$, the consumer will consume a smaller fraction of the additional current income. We can think of the term $\left( \frac{1}{1+\beta} \right)$ as representing the marginal propensity to consume (MPC) that we encountered in the Keynesian model.
6. (30 points). Consider the model of optimal investment discussed in class.
   a. Write the firm’s maximization problem.

   \[
   \begin{align*}
   \max_{I_1, I_2, L, K_2} & \quad V = A_1 K_1^{\theta} L_1^{1 - \theta} - w L_1 - I + \frac{A_2 K_2^{\theta} L_2^{1 - \theta} + (1 - \delta) K_2 - w L_2}{1 + r} \\
   \text{s.t.} & \quad K_2 = (1 - \delta) K_1 + I
   \end{align*}
   \]

   b. Explain in words what the firm wants to maximize.

   The firm wants to maximize the present value of the stream of dividends:

   \[ V = \pi_1 + \frac{\pi_2}{1 + r} \]

   c. According to this model, what should be stock price of the firm?

   The stock price should be the maximized value of \( V \).

   d. Derive the optimal investment condition and provide economic interpretation of it.

   Substituting the constraint into the objective gives

   \[
   \begin{align*}
   \max_{I_1, I_2, L, K_2} & \quad V = A_1 K_1^{\theta} L_1^{1 - \theta} - w L_1 - K_2 + (1 - \delta) K_1 + \frac{A_2 K_2^{\theta} L_2^{1 - \theta} + (1 - \delta) K_2 - w L_2}{1 + r} \\
   \text{F.O.C. for } K_2 : & \quad \frac{\partial V}{\partial K_2} = -1 + \frac{\theta A_2 K_2^{\theta - 1} L_2^{1 - \theta} + 1 - \delta}{1 + r} = 0
   \end{align*}
   \]

   At the optimum, the cost of increasing future capital by 1 unit must be equal to the benefit from that extra unit of capital. The benefit in the next period consists of the marginal product of capital and the non-depreciated value of the extra unit of capital. Dividing the next period’s benefit by \( 1 + r \) gives the present value of the benefit.

   We can rearrange the above to obtain \( \theta A_2 K_2^{\theta - 1} L_2^{1 - \theta} - \delta = r \). The left hand side is the net return on investment in physical capital and the right hand side is the net return on investment in the financial market.

   e. Illustrate graphically the impact of an increase in real interest rate on the demand for investment.
As real interest rate goes up, the optimal capital in the next period goes down, as shown in the figure above (from \( K_2 \) to \( K_2' \)). As a result, the optimal investment also goes down, since
\[
I = K_2 - (1 - \delta)K_1.
\]

f. Illustrate graphically the impact of future technological improvement on the demand for investment.

Notice that the marginal product curve shifts upward, so that for any given level of \( K_2 \) its marginal product increases. The optimal level of \( K_2 \) (and also of investment) will therefore increase.
7. (5 points). Draw a fully labeled diagram of the capital market for an open economy with trade deficit.

8. (5 points). Suppose the government increases its deficit. Illustrate graphically the impact of this event on the capital market. Show what happens to the equilibrium saving, investment, and trade deficit.
9. (5 points). Suppose that future productivity in the economy is expected to increase. Illustrate graphically the impact of this event on the capital market. Show what happens to the equilibrium saving, investment, and trade deficit.