Final Exam

Tuesday, December 16

1 hours, 30 minutes

Name: ___________________________________

Instructions

1. This is closed book, closed notes exam.
2. No calculators of any kind are allowed.
3. Show all the calculations.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

Good Luck 😊
1. (5 points). Suppose a variable $y_t$ is growing at constant rate $g$. Prove that the natural logarithm of that variable is a linear function of time. Indicate the slope and the intercept of the linear function.

If a variable is growing at constant rate $g$, then its value at time $t$ is given by

$$y_t = y_0 (1 + g)^t$$

Taking logs of the above:

$$\ln(y_t) = \ln(y_0) + t \ln(1 + g)$$

Which is a linear function of $t$, with slope $\ln(1 + g)$, and intercept $\ln(y_0)$.

2. (5 points). Let $x_t$, $y_t$ be two variables, and their growth rates are denoted $\hat{x}$, $\hat{y}$. Prove that the growth rate of the product $x \cdot y$ is approximately equal to the sum of the growth rates, for small growth rates. That is, prove that $\hat{x} \hat{y} \approx \hat{x} + \hat{y}$.

From the definition of the growth rate of the product

$$1 + \hat{x} \hat{y} = \frac{x_{t+1} y_{t+1}}{x_t y_t} = \frac{x_{t+1}}{x_t} \cdot \frac{y_{t+1}}{y_t} = (1 + \hat{x}) \cdot (1 + \hat{y})$$

Taking logs:

$$\ln(1 + \hat{x} \hat{y}) = \ln(1 + \hat{x}) + \ln(1 + \hat{y})$$

Using the theorem that $\ln(1 + g) \approx g$ for small $g$, gives the required result:

$$\hat{x} \hat{y} \approx \hat{x} + \hat{y}$$
3. (5 points). Suppose that in some economy the nominal GDP grows at 8%, the price level (GDP deflator) grows at 4%, and population grows at 2%. What is the approximate growth rate of Real GDP per capita? You are required to write the approximation formula used, and plug in the numbers.

We use the approximation formula that growth rate of a product is the approximately the sum of growth rates, and growth rate of a ratio is approximately the difference of the growth rates.

\[ \text{growth} \left( \frac{GDP}{P \cdot POP} \right) \approx \text{growth}(GDP) - \text{growth}(P) - \text{growth}(POP) = 8\% - 4\% - 2\% = 2\% \]

4. (15 points). Consider the Classical model studied in class, and briefly described as follows. The consumer derives utility from consumption \( C \) and leisure \( l \) according to \( U(C,l) = \alpha \ln C + (1 - \alpha) \ln l \). He is endowed with \( h \) hours which he can allocate between leisure and work \( L_S \). The real wage is \( w \). The consumer owns a firm and receives dividend income (profit) \( \pi \). The firm produces output \( Y \) using technology \( Y = AK^{\theta}L_D^{1-\theta} \), where \( A \) is productivity parameter (TFP), \( K \) is the capital owned by the firm, and \( L_D \) is labor employed by the firm. The government taxes labor income at the rate of \( t_w \) and dividend income at the rate of \( t_\pi \).

a. Write the consumer’s utility maximization problem.

Consumer’s problem
\[
\max_{C,l} \alpha \ln C + (1 - \alpha) \ln l
\]

subject to
\[
C = w(h - l)(1 - t_w) + \pi(1 - t_\pi)
\]
b. Derive the consumer's demand for consumption and leisure, and labor supply, using the results from Microfoundations. Explain your steps briefly.

We rewrite the budget constraint in the form of \( p_x x + p_y y = I \):

\[
C + w(1-t_w)l = wh(1-t_w) + \pi (1-t_{\pi})
\]

**Explanation:** From Microfoundations, we know that consumers with these Cobb-Douglas preferences, spend a fraction \( \alpha \) of their income on \( C \) and a fraction \( 1-\alpha \) of their income on \( l \). Thus, the demand is:

Demand for consumption: \( C^* = \alpha[wh(1-t_w) + \pi (1-t_{\pi})] \)

Demand for leisure: \( l^* = \frac{(1-\alpha)[wh(1-t_w) + \pi (1-t_{\pi})]}{w(1-t_w)} = (1-\alpha) \left( h + \frac{\pi (1-t_{\pi})}{w (1-t_w)} \right) \)

c. Suppose that initially \( t_w = t_{\pi} = t = 30\% \), and equilibrium employment and equilibrium output were \( L^* = 42 \), \( Y^* = 1200 \). The government wants to stimulate the economy by lowering the tax rate to \( t = 20\% \). Find the new equilibrium employment and equilibrium output. Explain your answer briefly.

\( L^* = 42 \), \( Y^* = 1200 \)

In a model with the same tax rate on both types of income, the government cannot affect the equilibrium employment or output by changing the common tax rate. The only effect of changes in taxes is on the division of output between private and government consumption.
5. (20 points). Consider the capital market in a closed economy, with supply of capital (private saving) given by \( S_p = 2r \) and private demand for investment \( I = 10 - 0.5r \), both are functions of the real interest rate \( r \) (in percent).

a. Suppose that government budget is balanced. Solve for equilibrium in the capital market (find equilibrium real interest rate, private saving, government saving, and investment).

In a closed economy, the saving and investment equation is:

\[
S_p + S_G = I \\
2r + 0 = 10 - 0.5r \\
2.5r = 10 \\
r^* = 4 \\
S_p^* = 2 \cdot 4 = 8 \\
S_G = 0 \text{ (given)} \\
I^* = 10 - 0.5 \cdot 4 = 8
\]

b. Suppose that the government runs a budget deficit at the size of 2.5, in an attempt to stimulate the economy. Solve for equilibrium in the capital market (find equilibrium real interest rate, private saving, government saving, and investment).

\[
S_p + S_G = I \\
2r - 2.5 = 10 - 0.5r \\
2.5r = 12.5 \\
r^* = 5 \\
S_p^* = 2 \cdot 5 = 10 \\
S_G = -2.5 \text{ (given)} \\
I^* = 10 - 0.5 \cdot 5 = 7.5
\]
c. The last section illustrates a mechanism of crowding out. True/false, circle the correct answer and explain briefly, based on your results in the last section.

Indeed, higher deficit means that the government needs to borrow more in the capital market, and this increases the real interest rate (from 4% to 5%). This lowers private investment (from 8 to 7.5), which is crowding out of private investment. Moreover, the rise in private saving means decline in private consumption, ceteris paribus, which is crowding out of private consumption.

d. Draw a fully labeled graph of the capital market, demonstrating the initial equilibrium in part a, and the event described in part b.
6. (20 points). Consider the two-period model of consumption and saving discussed in class. There are $N$ identical consumers that live for two periods (1 and 2) and derive utility from consumption $c_1$ and $c_2$ in the two periods: $U(c_1, c_2)$. Consumers receive income $y_1$ and $y_2$ in the two periods and pay a lump sum tax $t_1$ and $t_2$ to the government. The consumers decide how much to consume in each period and how much to save in the first period. We denote the saving in the first period by $s$. Consumers can borrow and lend at real interest rate $r$, which is assumed exogenously given. Thus the budget constraints in the two periods are

$$BC_1 : \quad c_1 + s = y_1 - t_1$$
$$BC_2 : \quad c_2 = y_2 - t_2 + (1 + r) s$$

The government collects tax revenues $T_1 = N \cdot t_1$ and $T_2 = N \cdot t_2$, and spends $G_1$ and $G_2$ in the two periods. The government can borrow and lend at real interest rate $r$ with the constraint that the present value of spending = present value of taxes

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r}$$

a. Suppose that the real interest rate is $r = 10\%$ and the government lowers the taxes by 200. In the second period, the government will have to increase/decrease taxes (circle the correct answer) by ____220_______ (calculate the necessary increase or decrease).

Plugging the current change in taxes into the government budget constraint, and solving for the necessary future tax change:

$$T_1 - 200 + \frac{T_2 + \Delta T_2}{1+r} = T_1 + \frac{T_2}{1+r}$$

$$-200 + \frac{\Delta T_2}{1+r} = 0$$

$$\Delta T_2 = 200(1+r) = 200 \cdot 1.1 = 220$$
b. Suppose that the consumer’s utility is \( U(c_1, c_2) = \ln(c_1) + \beta \ln(c_2) \). Write the consumer’s demand for consumption in both periods and his supply of saving.

Writing the consumer’s problem with the lifetime budget constraint helps for this section.

\[
\begin{align*}
\max_{c_1, c_2} & \quad \ln(c_1) + \beta \ln(c_2) \\
\text{s.t.} & \quad c_1 + \frac{c_2}{1+r} = y_1 - t_1 + \frac{y_2 - t_2}{1+r}
\end{align*}
\]

Now we can see that since the preferences are of the Cobb-Douglas form, the consumer will spend a fixed fraction of his lifetime income on \( c_1 \) and \( c_2 \):

\[
\begin{align*}
c_1^* &= \left(\frac{1}{1+\beta}\right) \left( y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right) \\
c_2^* &= \left(\frac{\beta}{1+\beta}\right) \left( y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right) (1+r) = \left(\frac{\beta}{1+\beta}\right) \left( (y_1 - t_1)(1+r) + y_2 - t_2 \right)
\end{align*}
\]

The saving, from the first budget constraint

\[
s^* = y_1 - t_1 - \left(\frac{1}{1+\beta}\right) \left[ y_1 - t_1 + \frac{y_2 - t_2}{1+r} \right]_{c_1^*}
\]
c. Consumers with higher \( \beta \) are more likely to be borrowers. True/false, circle the correct answer and provide a proof and economic intuition for your answer.

From the optimal saving, derived above, we have:

\[
\frac{\partial s^*}{\partial \beta} = \frac{we}{(1 + \beta)^2} > 0
\]

where \( we = y_1 - t_1 + \frac{y_2 - t_2}{1 + r} \).

Thus, optimal saving is increasing in \( \beta \), so consumers with higher \( \beta \) will save more, ceteris paribus, and are more likely to be lenders, not borrowers.

Intuitively, \( \beta \) is the weight on utility from future consumption. Higher \( \beta \) means that future consumption is more important, and consumers will tend to save more.

d. Suppose that income in the first period increases by $1000. Find the resulting change in consumption of both periods, and the change in saving, if the real interest rate is \( r = 7\% \) and \( \beta = 1 \).

\[
\Delta c_1^* = \left( \frac{1}{1 + \beta} \right) \Delta y_1 = \frac{1}{2} \cdot 1000 = 500
\]

\[
\Delta c_2^* = \left( \frac{\beta}{1 + \beta} \right) (1 + r) \Delta y_1 = 0.5 \cdot 1000 \cdot 1.07 = 535
\]

\[
\Delta s^* = \left( 1 - \frac{1}{1 + \beta} \right) \Delta y_1 = 0.5 \cdot 1000 = 500
\]

or

\[
\frac{\Delta s^*}{500} + \frac{\Delta c_1^*}{1000} = \frac{\Delta y_1}{1000}
\]
7. (15 points). Consider the quantity theory of money equation: \( MV = PY \).

a. Write this equation in approximate growth rates.

\[ \dot{M} + \dot{V} = \dot{P} + \dot{Y} \]

b. Suppose that velocity is constant, the growth rate of real GDP is 5% and the central bank wants to achieve inflation of 2%. What is the required growth of the money supply?

\[ \dot{M} + \dot{V} = \dot{P} + \dot{Y} \]
\[ ? \quad 0\% \quad 2\% \quad 5\% \]
\[ \Rightarrow \dot{M} = 7\% \]

c. Now suppose that velocity growth is decreasing in the growth rate of money: 
\( \dot{V} = -0.3\dot{M} \), other things being the same as in the previous section. How would your answer the part b change?

\[ \dot{M} + \dot{V} = \dot{P} + \dot{Y} \]
\[ ? \quad -0.3\dot{M} \quad 2\% \quad 5\% \]
\[ \dot{M} - 0.3\dot{M} = 7\% \]
\[ 0.7\dot{M} = 7\% \]
\[ \Rightarrow \dot{M} = 10\% \]
8. (20 points). Suppose that the public wants to hold currency/deposit ratio of \( cd = 0.2 \), and the required reserve/deposit ratio is \( rd = 0.4 \). The initial consolidated balance sheet of commercial banks is:

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = ? )</td>
<td>( D = 100 )</td>
</tr>
<tr>
<td>( B_G = 15 )</td>
<td></td>
</tr>
<tr>
<td>( L = ? )</td>
<td></td>
</tr>
</tbody>
</table>

a. Find the monetary base, the money supply and the money multiplier in this economy, and complete the missing values in the above balance sheet.

\[
\begin{align*}
CU &= cd \cdot D = 0.2 \cdot 100 = 20 \\
R &= rd \cdot D = 0.4 \cdot 100 = 40 \\
MB &= CU + R = 20 + 40 = 60 \\
M &= CU + D = 20 + 100 = 120 \quad (or \quad M = mm \cdot MB) \\
\frac{mm}{cd + rd} &= \frac{0.2 + 1}{0.2 + 0.4} = 2
\end{align*}
\]

Since total assets are 100, and \( R = 40, B_G = 15 \), then loans are

\[
L = 100 - R - B_G = 45
\]
b. Suppose that the central bank performs an open market operation and buys government bonds from the commercial banks at the amount of 6. Find the new monetary base, the money supply and show the new balance sheet of the commercial banks.

\[ MB = 66 \]
\[ M = mm \cdot MB = 2 \cdot 66 = 132 \]
\[ D = \left( \frac{1}{rd + cd} \right) \cdot MB = \left( \frac{1}{0.4 + 0.2} \right) \cdot 66 = 110 \]
\[ R = \left( \frac{rd}{rd + cd} \right) \cdot MB = \left( \frac{0.4}{0.4 + 0.2} \right) \cdot 66 = 44 \]
\[ CU = \left( \frac{cd}{rd + cd} \right) \cdot MB = \left( \frac{0.2}{0.4 + 0.2} \right) \cdot 66 = 22 \]

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = 44 )</td>
<td>( D = 110 )</td>
</tr>
<tr>
<td>( B_G = 9 )</td>
<td>( D = 110 )</td>
</tr>
<tr>
<td>( L = 57 )</td>
<td>( D = 110 )</td>
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<td>( 110 )</td>
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