MATH 245: Differential Equations and Linear Algebra

# MIDTERM I

Fall 2009

NAME : ANSWER KEY

*NOTE*: There are 5 problems on this midterm (total of 6 pages). Use of calculators to check your work is permitted; however, in order to receive full credit for any problem, you must show work leading to your answer. You have 50 minutes to complete this test.

| Problem | Possible points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

| 20+ | 30+ | 40+ | 50+ | 60+ | 70+ | 80+ | 90+ |
|---|---|---|---|---|---|---|---|
| 4 | 4 | 7 | 2 | 4 | 3 | 2 | 1 |

D | C | B | A

40 | 32 | 24 | 16

35 | 55 | 70

Flawless Bonus

**Problem 1.** (20pts) Solve the initial value problem.

$$y' + 2xy = x, \qquad y(0) = -2$$

IF $= e^{x^2}$

$$(ye^{x^2})' = xe^{x^2}$$

$$ye^{x^2} = \frac{1}{2}e^{x^2} + C$$

$$y = \frac{1}{2} + C_1 e^{-x^2}$$

$$-2 = \frac{1}{2} + C_1 \Rightarrow C_1 = -\frac{5}{2}$$

$$\therefore y(x) = \frac{1 - 5e^{-x^2}}{2} \quad //$$

**Problem 2.** (20pts) Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10 seconds later the boat has slowed to 20 ft/s. Assume a linear resistance, so that $dv/dt = -kv$ for some constant $k > 0$. How far will the boat coast in all?

$$\int \frac{dv}{v} = \int -k\,dt \Rightarrow \ln v = -kt + C_1'$$

$$\Rightarrow v = C \cdot e^{-kt}$$

$$v(0) = 40 \Rightarrow C = 40$$

$$v(10) = 20 \Rightarrow 20 = 40\, e^{-10k}$$

$$\Rightarrow e^{10k} = 2 \Rightarrow k = \frac{1}{10} \ln 2$$

$$\frac{dx}{dt} = 40\, e^{-kt}$$

$$x(t) = C'' - \frac{40}{k} e^{-kt}$$

$$x(0) = 0 \Rightarrow C'' = \frac{40}{k}$$

$$\therefore x(t) = \frac{40}{k} - \frac{40}{k} e^{-kt} \longrightarrow \frac{40}{k} \text{ as } t \to \infty$$

<u>coast</u> $\frac{40}{\frac{1}{10}\ln 2} = \frac{400}{\ln 2}$ ft in all

**Problem 3.** (20pts) Show that the solution curves of the differential equation

$$\frac{dy}{dx} = -\frac{y(2x^3 - y^3)}{x(2y^3 - x^3)}$$

are of the form $x^3 + y^3 = 3Cxy$.

$$y = vx \qquad y' = v'x + v = -\frac{v(2-v^3)}{2v^3-1}$$

$$v'x = \frac{-2v + v^4 - 2v^4 + v}{2v^3-1} = \frac{-v^4 - v}{2v^3-1}$$

$$\int \frac{2v^3-1}{v^4+v}\,dv = \int -\frac{1}{x}\,dx \qquad v^4+v = v(v+1)(v^2-v+1)$$

$$\frac{2v^3-1}{v(v+1)(v^2-v+1)} = \frac{A}{v} + \frac{B}{v+1} + \frac{Cv+D}{v^2-v+1}$$

$$A = -1, \quad B = \frac{2(-1)^3-1}{(-1)(3)} = 1$$

$$2v^3 - 1 = -(v+1)(v^2-v+1) + v(v^2-v+1) + (Cv+D)(v^2+v)$$

$$= -\cancel{v^3} - 1 + \cancel{v^3} - v^2 + v + Cv^3 + Cv^2 + Dv^2 + Dv$$

$$2 = C$$
$$0 = -1 + C + D \quad \Rightarrow \quad D = -1$$

$$\int \left(\frac{1}{v+1} - \frac{1}{v} + \frac{2v-1}{v^2-v+1}\right) dv = -\ln x + C_1'$$

$$\cancel{\ln}\left|\frac{(v+1)(v^2-v+1)}{v}\right| = \frac{C}{x}$$

$$v^3 + 1 = \frac{Cv}{x} \quad \Rightarrow \quad y^3 + x^3 = Cyx \;//$$

**Problem 4.** (20pts) Apply Euler's method with a step size $h = 0.1$ to approximate the solution to the following initial value problem on the interval $[0, .5]$.

$$y' = -2xy, \qquad y(0) = 2$$

$$y_{n+1} = y_n + 0.1(-2x_n y_n)$$

| $x_n$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|---|
| $y_n$ | 2.0 | 2.0 | 1.96 | 1.8816 | 1.7687 | 1.6272 |

$y_0 = 2$

$y_1 = 2.0 + 0.1(0) = 2.0$

$y_2 = 2.0 + 0.1(-0.2)(2.0) = 1.96$

$y_3 = 1.96 + 0.1(-0.4)(1.96) = 1.8816$

$y_4 = 1.8816 + 0.1(-0.6)(1.8816) = 1.768704$

$y_5 = 1.768704 + 0.1(-0.8)(1.768704) = 1.62723$

**Problem 5.** (20pts) Find the general solution to the following differential equation.

$$y^{(4)} - 3y'' - 4y = 2x^2 + e^{2x}$$

$r^4 - 3r^2 - 4 = 0$

$(r^2 - 4)(r^2 + 1) = 0$

$r = \pm 2, \pm i$ $\quad e^{2x}, e^{-2x}, \cos x, \sin x$

$y_p = Ax^2 + Bx + C + Dxe^{2x}$

$y_p' = 2Ax + B + (D + 2Dx)e^{2x}$

$y_p'' = 2A + (2D + 2D + 4Dx)e^{2x}$

$= 2A + (4D + 4Dx)e^{2x}$

$y_p''' = (4D + 8D + 8Dx)e^{2x} = (12D + 8Dx)e^{2x}$

$y_p^{(4)} = (8D + 24D + 16Dx)e^{2x} = (32D + 16Dx)e^{2x}$

$e^{2x}(32D + \cancel{16Dx} \cancel{+ (12D} \cancel{+ 12Dx} - \cancel{4Dx}) - 3(2A) - 4(Ax^2 + Bx + C) = 2x^2 + e^{2x}$

$-4A = 2 \Rightarrow A = -1/2$

$-4B = 0 \Rightarrow B = 0$

$-6A - 4C = 0 \Rightarrow C = 3/4$

$20D = 1 \Rightarrow D = \frac{1}{20}$

$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x + \frac{3}{4} - \frac{x^2}{2} + \frac{xe^{2x}}{20}$