

LAB #1: EFFECT OF ROOTS ON THE TIME AND FREQUENCY RESPONSES

(Updated Dec. 23, 2002)

Objective:

To investigate the effect of root location in the complex plane upon the *step* and *frequency responses* of 1st-order and 2nd-order circuits. To use op amps to control root location and observe its impact upon stability. To see how instability can be exploited to generate oscillations. To compare *experimental* results with *theory* and *PSpice simulations*, and account for possible differences. To review *Bode plots*.

Components:

1 × 741 op amp, 2 × 1N4148 diodes, 1 × 2N2222 *npn* BJT, 2 × 22 nF capacitors, 2 × 0.1 μF capacitors, 1 × 10-kΩ potentiometer, and resistors: 1 × 100 Ω, 1 × 1.5 kΩ, 1 × 3.3 kΩ, 1 × 6.8 kΩ, 4 × 10 kΩ, and 1 × 33 kΩ (all 5%, ¼ W).

Instrumentation:

A dual ±15-V regulated power supply, a digital multi-meter (DMM), a signal generator (sine wave and square wave), and a dual-trace oscilloscope.

References:

1. Sergio Franco, *Electric Circuits Fundamentals*, Oxford University Press, 1995 (Chs. 8, 9, and 14).
2. Sergio Franco, *Design with Op Amps and Analog ICs*, 3rd Edition, McGraw-Hill, 2002 (Chs. 3 & 10)

PART I – THEORETICAL BACKGROUND

The behavior of a circuit is uniquely determined by its transfer function $H(s) = V_o(s)/V_i(s)$, where $V_i(s)$ and $V_o(s)$ are the Laplace transforms of the circuit's input $v_i(t)$ and output $v_o(t)$, and s is the *complex frequency*, in complex Np/sec. For physical circuits, $H(s)$ takes on the form of a *rational function* of s , that is, $H(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are *polynomials* of degrees m and $n \geq m$, respectively. In this lab we are particularly interested in 1st-order ($n = 1$) and 2nd-order ($n = 2$) circuits. The roots of $N(s) = 0$ and those of $D(s) = 0$ are called, respectively, the *zeros* and the *poles* of $H(s)$. Roots can be real or complex; when complex, the roots of a physical system always occur in *conjugate pairs*.

The response of a circuit to an external impulse, aptly called the *impulse response*, offers an indication of how the circuit does on its own, using the energy stored internally in its energy-storage elements as a consequence of the applied impulse. Such a response is found mathematically as $h_o(t) = L^{-1}\{H(s)\}$, where L^{-1} stands for *inverse Laplace transform*. We identify two significant cases:

- $H(s)$ contains a real pole at $s = \sigma + j0$, or

$$H(s) = \frac{A}{s - \sigma}$$

where A is a suitable scaling factor, $j^2 = -1$, and σ is the *Neper frequency*, in Np/s. The impulse response is found to be

$$h_o(t) = L^{-1}\{H(s)\} = Ae^{\sigma t}u(t) \tag{1}$$

where $u(t)$ is the unit step function ($u = 0$ for $t < 0$, $u = 1$ for $t > 0$). We note that this response is (a)

an *exponential decay* if $\sigma < 0$, (b) a constant or *sustained* response if $\sigma = 0$, or (c) a *diverging exponential* if $\sigma > 0$. A diverging response indicates an *unstable system*. Such a response will grow until limited by some inherent circuit nonlinearity, such as op amp saturation. It is apparent that for a 1st-order system to be stable, its *pole must lie on the negative real axis*, where $\sigma < 0$.

- $H(s)$ contains a complex-conjugate pole pair at $s = \sigma \pm j\omega$, indicating that it can be expressed as

$$H(s) = \frac{A}{[s - (\sigma + j\omega)] \times [s - (\sigma - j\omega)]}$$

where A is a suitable scaling factor, σ is the *Neper frequency* (in Np/s), and ω is the *angular frequency* (in rad/s). The impulse response is now found to be

$$h_o(t) = L^{-1}\{H(s)\} = Be^{\sigma t}u(t)\cos(\omega t + \phi) \quad (2)$$

where B and ϕ are suitable constants. We note that the impulse response is (a) an *exponentially decaying* or *damped sinusoid* if $\sigma < 0$, (b) a *sustained* or *undamped sinusoid* if $\sigma = 0$, and (c) a *diverging sinusoid* if $\sigma > 0$. A diverging sinusoid indicates again an *unstable system*. Such a response will grow until limited by some inherent circuit nonlinearity, such as op amp saturation. It is apparent that for a 2nd-order system to be stable, its *pole pair must lie in the left of the complex plane*, where $\sigma < 0$. A circuit whose pole pair lies right on the imaginary axis, for which $\sigma = 0$, is aptly called an *oscillator*. In this laboratory we shall use an op amp to control the location of poles in the complex plane.

1st-Order Circuit

To develop physical insight for the case of a 1st-order circuit, consider the simple R - C network of Fig. 1. We know from basic circuit theory that if C is initially charged at some voltage $v(0) \neq 0$, its energy will be dissipated by R , causing C to discharge exponentially with the time constant $\tau = RC$, according to

$$v(t \geq 0) = v(0)e^{-t/RC} \quad (3a)$$

This is also known as the *natural* or *source-free* response of the circuit. As we know, the R - C network exhibits a pole on the *negative real axis*,

$$\sigma = -\frac{1}{\tau} = -\frac{1}{RC} \quad (3b)$$

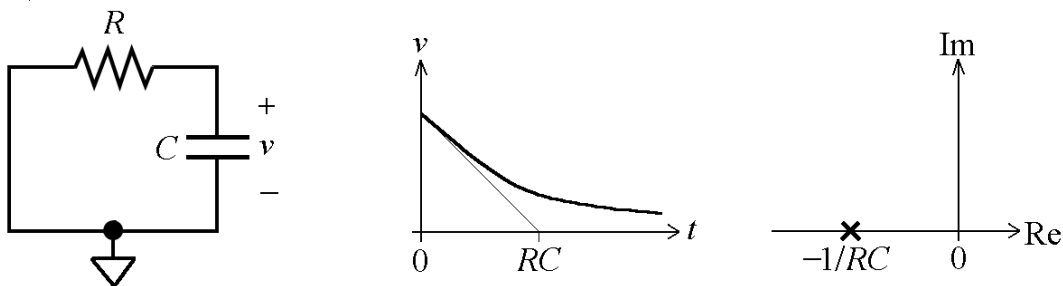


Fig. 1 – Basic R - C circuit, its natural response, and s -plane location of its pole.

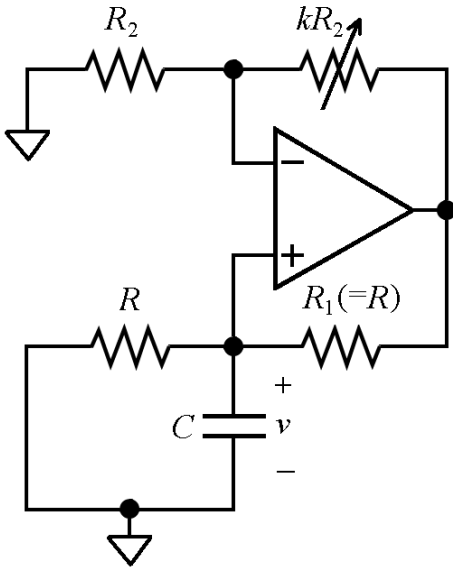


Fig. 2 – Using an op amp for single-pole control.

Consider now the addition of a noninverting op amp to amplify the capacitor voltage v and then re-inject energy into the R - C network via R_1 , as shown in Fig. 2. With the component designations shown, the op amp amplifies v by $1 + kR_2/R_2 = 1 + k$. The re-injected energy will make up for some of the energy dissipated by R , effectively resulting in a longer time constant τ , or, equivalently, in a smaller σ . We still have a pole lying on the *negative real axis*, but closer to the origin. If the re-injected energy happens to *equal* the dissipated energy, then the capacitor voltage will remain constant, indicating $\tau = \infty$, or, equivalently, $\sigma = 0$. Clearly, the pole now lies *right at the origin* of the complex plane. If, on the other hand, the re-injected energy happens to *exceed* the dissipated energy, then the capacitor voltage will diverge, indicating $\tau < 0$, or, equivalently, $\sigma > 0$. We now have a pole in the *right-half of the complex plane*, and the circuit is unstable.

The circuit can be analyzed quantitatively by observing that the op amp and associated resistors form a *negative resistance converter* synthesizing a resistance whose value, with the component designations shown, is $-R_1(R_2/kR_2) = -R/k$. As depicted in Fig. 3a, the equivalent resistance seen by the capacitor is then $R_{eq} = R/(-R/k) = R/(1 - k)$, indicating a time constant $\tau = R_{eq}C = RC/(1 - k)$, or a pole at

$$\sigma = -\frac{1}{\tau} = \frac{k-1}{RC} \quad (4a)$$

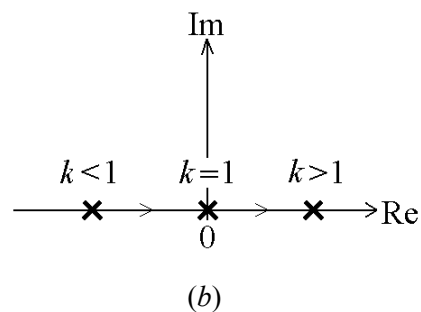
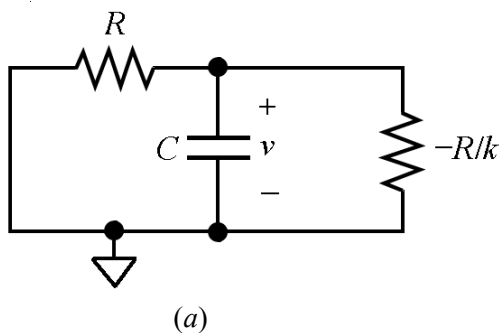


Fig. 3 – Using the concept of *negative resistance* to illustrate *pole location* control.

Consequently, the natural response is

$$v(t \geq 0) = v(0)e^{(k-1)t/RC} \quad (4b)$$

Depending on whether $k < 1$, $k = 1$, or $k > 1$, this response *decays*, remains *constant*, or *diverges*, respectively. Figure 3b shows the *root locus*, that is, the trajectory of the pole as we vary k .

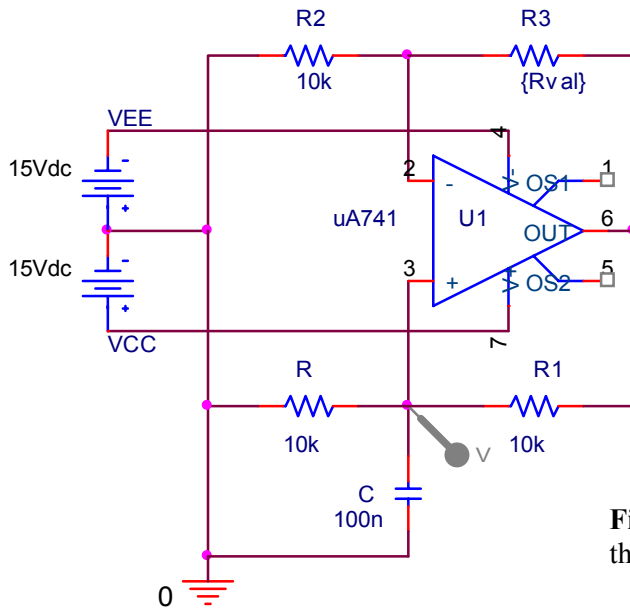


Fig. 4 – PSpice circuit to visualize the three response types of a 1st-order system.

The three response types are readily visualized via PSpice. The required circuit is shown in Fig. 4, where the initial condition $v(0) = 1$ V has been imposed. The responses are depicted in Fig. 5. You can duplicate this PSpice example on your own by downloading its appropriate files from the Web. To this end, go to <http://online.sfsu.edu/~sfranco/CoursesAndLabs/Labs/445Labs.html>, and once there, click on **PSpice Examples**. Then, follow the instructions contained in the **Readme** file. In fact, you are urged to run this example and modify the simulation profile by extending the run time by a few more ms. This will allow you to observe that the diverging response keeps increasing until the op amp saturates, at which point the response stabilizes at half the output saturation voltage of the op amp. Can you explain why?

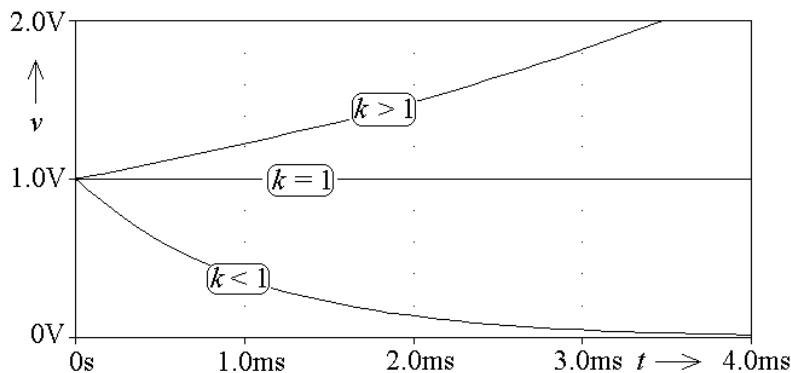


Fig. 5 – Responses for the PSpice circuit of Fig. 4.

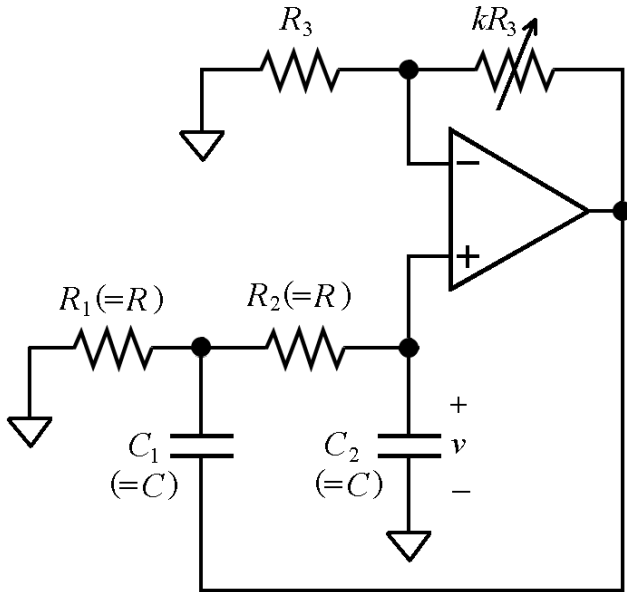


Fig. 6– Using an op amp for pole-pair control.

2nd-Order Circuit

Proceeding along a similar line of reasoning, we can develop physical insight also for the 2nd-order circuit of Fig. 6. This circuit consists of two R - C stages in cascade (R_1 - C_1 and R_2 - C_2), along with a non-inverting amplifier designed to amplify the voltage v across C_2 by $1 + kR_3/R_3 = 1 + k$, and then re-inject energy upstream of the R_2 - C_2 stage via C_1 . By adjusting the amplifier's gain we can control the amount of re-injected energy and have the natural response take on the form of a *damped*, a *sustained*, or a *diverging* sinusoid, respectively. As we shall see below, the three cases correspond, respectively, to $k < 2$, $k = 2$, and $k > 2$.

We can again visualize the three cases via PSpice. The necessary circuit is shown in Fig. 7, where we are imposing initial voltages of 1 V across C_1 and 0 V across C_2 . The three response types are

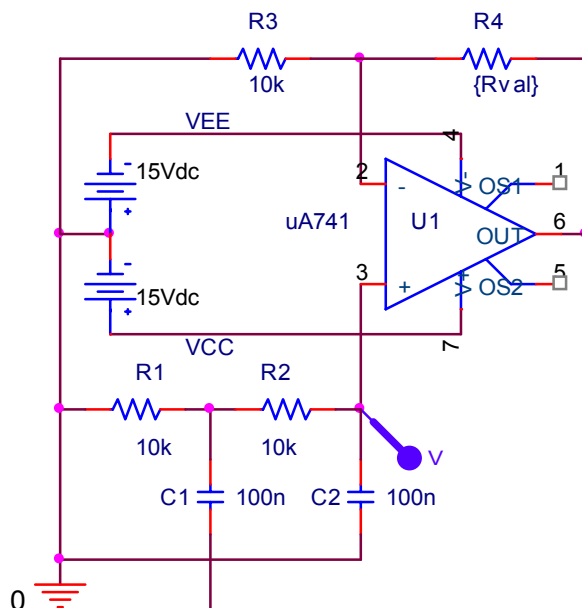


Fig. 7 – PSpice circuit to visualize the three response types of a 1st-order system.

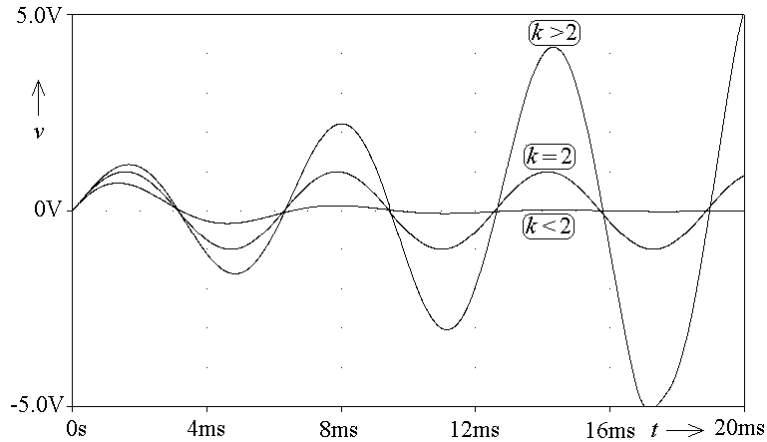


Fig. 8 – Responses for the PSpice circuit of Fig. 7.

shown in Fig. 8. You are again encouraged to replicate this simulation on your own, but with an extended run time in order to observe the effect of op amp saturation upon the diverging sinusoid. Can you justify quantitatively what you see?

The effect of k upon the pole location can again be visualized via the *root locus*, representing the trajectory described by the *pole pair* as k is varied from $k = 0$ to $k > 2$. This locus is shown in Fig. 9a.

Frequency Response of Stable Circuits:

Stable circuits are often characterized via their *frequency response*, that is, the manner in which the circuit processes the *amplitude* and *phase* of an ac input as a function of frequency. Systems theory tells us that once the transfer function $H(s)$ is known, such a response is obtained by letting $s \rightarrow j\omega$ (or $s \rightarrow j2\pi f$) in the expression for $H(s)$. Conversely, the frequency response can be found experimentally as

$$|H(jf)| = \frac{V_{om}}{V_{in}} \quad \angle H(jf) = \theta_o - \theta_i \quad (5)$$

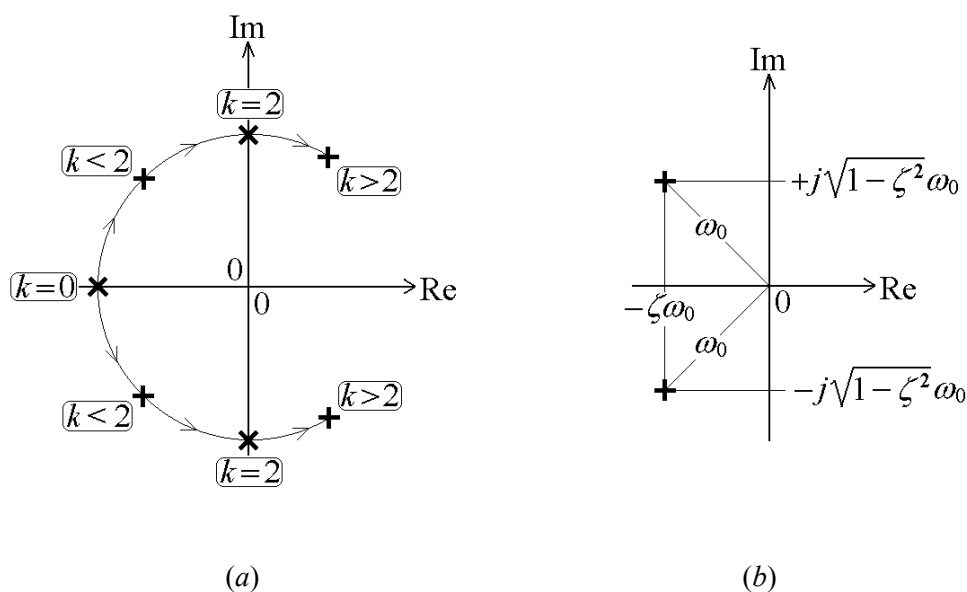


Fig. 9 – (a) Root locus as a function of k , and (b) pole pair for the case of a damped response.

where $|H(jf)|$ and $\angle H(jf)$ represent *magnitude* and *phase* of $H(jf)$, and V_{im} , V_{om} , θ_i and θ_o are the amplitudes and phases of the *input* and *output signals*, $v_i = V_{im}\cos(2\pi ft + \theta_i)$ and $v_o = V_{om}\cos(2\pi ft + \theta_o)$. As we know, the plots of $|H(jf)|$ (in decibels) and of $\angle H(jf)$ (in degrees) versus f (in decades) are referred to as the *Bode Plots* of H .

PART II – EXPERIMENTAL PART

Most measurements shall be performed using the circuits of Figs. 10 and 12, which each lab group should assemble simultaneously in separate areas of the proto-board before coming to the lab. This will allow using the allotted lab time efficiently, primarily to perform the required measurements and observations. Refer to the Appendix for useful tips on how to construct op amp circuits. In particular, use two 0.1- μ F capacitors to bypass the ± 15 -V power supplies, and always turn off power before making any changes in a circuit. Failure to do so may destroy the op amps as well as other delicate components.

Henceforth, steps shall be identified by letters as follows: **P** for prelab work, **C** for calculations, **M** for measurements, and **S** for SPICE simulation.

1st-Order Circuits:

PC1: Show that for the circuit of Fig. 10 we have

$$H(s) = \frac{H_0}{1 + s/\omega_0} \tag{6a}$$

$$H_0 = \frac{1+k}{1-k} \qquad \omega_0 = \frac{1-k}{RC} \tag{6b}$$

As we know, this system has a pole at

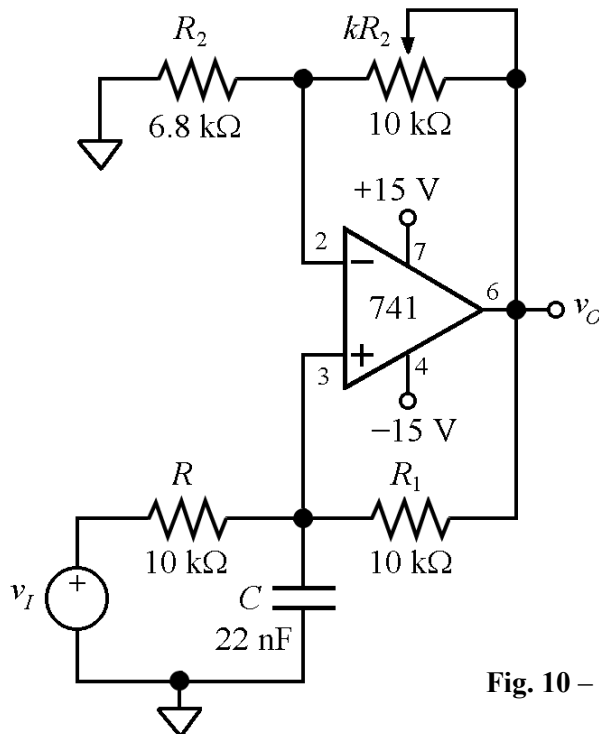


Fig. 10 – Circuit to investigate 1st-order frequency responses.

$$s = -\omega_0 = \frac{k-1}{RC} \quad (7)$$

MC2: With power off, assemble the circuit of Fig. 10 and set the wiper *all the way to the left* to make $kR_2 = 0$, or $k = 0$. Also, while monitoring v_I with Ch. 1 of the oscilloscope set on DC, adjust the waveform generator so that v_I is a *sine wave* with 0-V DC and a *peak-to-peak amplitude* of 2 V.

Next, apply power, and while monitoring v_O with Ch. 2 of the oscilloscope (also set on DC), measure the *magnitude response* as per Eq. (5) as a function of frequency. To this end, first find the -3 -db frequency $f_{-3\text{db}}$ of the circuit, that is, the frequency at which V_{om} drops to 70.7% of its low-frequency value; then, while keeping V_{im} constant, vary frequency and measure V_{om} at a few other significant frequency points using your judgment to economize on the number of measurements while still extracting all significant information.

Finally, turn power off, pull R out of the circuit, measure its value with the ohmmeter, and find the actual value of C from $f_{-3\text{db}} = 1/2\pi RC$. How does it compare with the nominal value?

C3: Using the data of Step M2, plot $|H(jf)|$ (in decibels) versus f (in decades). Hence, verify that within measurement errors, the low- and high-frequency asymptotes meet at $f_{-3\text{db}}$, and that for $f \gg f_{-3\text{db}}$ the *gain-bandwidth product* $\text{GBP} = |H(f)| \times f$ is constant.

MC4: Repeat Steps MC2 and C3, but with the potentiometer set for $kR_2 = R_2/2$, or $k = 0.5$. (For an accurate setting, you may want to use the ohmmeter to measure both R_2 and kR_2 ; pull your resistors out of the circuit as you measure them!). How does the Bode plot change as you change k from 0 to 0.5?

S5: Using the component values measured in Step MC2, perform a PSpice simulation of the circuit of Fig. 10 for $k = 0, 0.5, 0.9$, and 1.0. Display the magnitude Bode plots, and sketch and label the complex-plane locations of the corresponding poles.

M6: We are now going to use the circuit of Fig. 11 to observe the *time-domain* behavior. Here, the voltage divider made up of the 1.5-k Ω and 100- Ω resistances is designed to force a DC input of about 1 V to the RC network, and the BJT switch is used to periodically reset the capacitor voltage to 0 V in order to ensure a repetitive trace on the oscilloscope. Thus, proceed as follows:

- Assemble the circuit with power off, and while monitoring v_{Reset} with Ch.1 of the oscilloscope set on DC, adjust the waveform generator so that v_{Reset} is a *pulse train* alternating between -1 V and $+5$ V with an initial frequency of about 1 kHz and a pulse width of several microseconds (as you progress through the experiment, you may find it necessary to alter these initial recommended setting for best visualization on the scope.)
- Next, while monitoring v_O with Ch.2 of the oscilloscope (also set on DC), slowly vary the wiper from the left to the right to see how the response changes from a *converging exponential* ($k < 1$), to a *ramp* ($k = 1$) to a *diverging exponential* ($k > 1$), in a manner similar to Fig. 5. Show your results to the lab instructor.
- Record the waveform of v_O for *three* different values of k , namely, $k < 1$, $k = 1$, and $k > 1$. The first and last values of k are of your own choice, so make sure that once you are finished, you measure the corresponding values of kR_2 and R_2 in order to compute the actual values of k (pull them out of the circuit to measure them!)
- For the cases $k = 1$ and $k > 1$, record v_O also with the frequency of v_{Reset} adjusted until the op amp *just begins* to saturate. Comment on all of your results

S7: Simulate the circuit Step M5 via PSpice for each of the three chosen values of k . Hence, compare with Step M6, and account for possible differences.

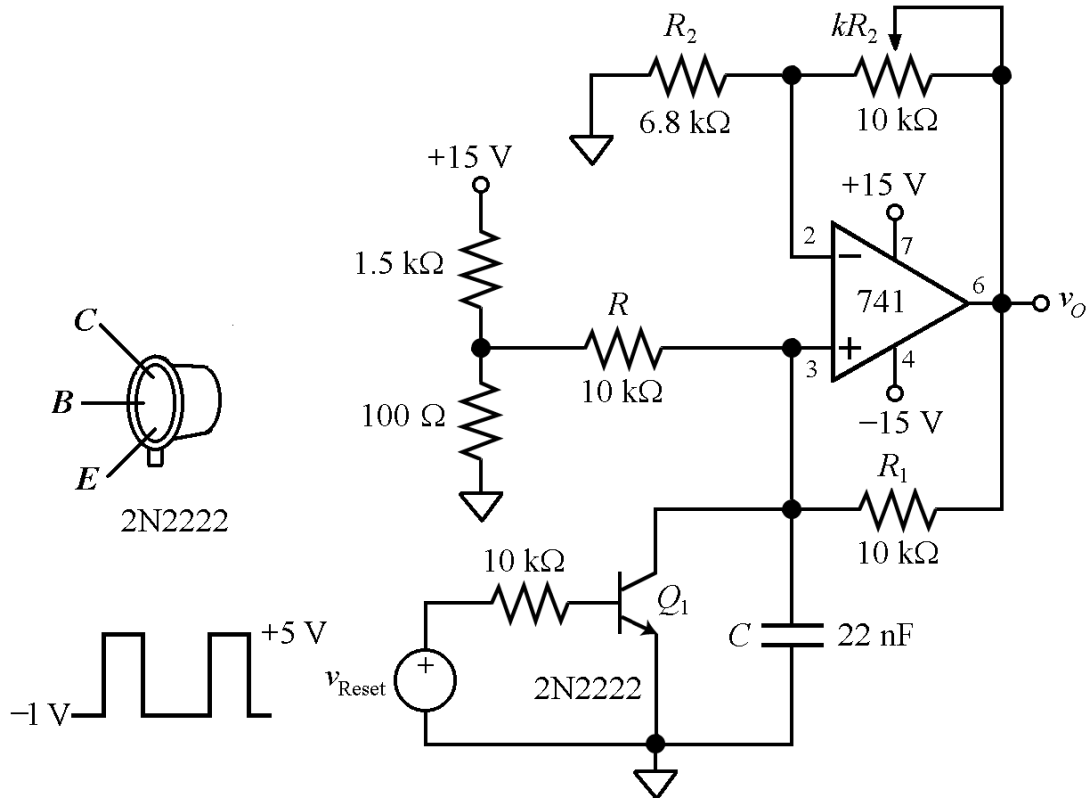


Fig. 11 – Circuit to investigate 1st-order transients.

C8: For *each* of the *three* values of k above: (a) develop a *mathematical expression* for $v_o(t)$, and (b) sketch and label on graph paper the *s-plane location* of the corresponding pole.

2nd-Order Circuits:

PC9: Show that for the circuit of Fig. 12 we have

$$H(s) = \frac{H_0}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1} \quad (8a)$$

$$H_0 = 1 + k \quad \omega_0 = \frac{1}{RC} \quad \zeta = 1 - 0.5k \quad (8b)$$

where ω_0 is the *undamped natural frequency* and ζ is the *damping ratio*. Hence, show that for $0 < \zeta < 1$ the poles of $H(s)$ are *complex conjugate* and occur at

$$s_{1,2} = \sigma \pm j\omega = -\zeta\omega_0 \pm j\sqrt{1 - \zeta^2} \omega_0 \quad (9)$$

Figure 9b depicts the pole location for this particular but important case.

MC10: With power off, assemble the circuit of Fig. 12 and adjust the wiper for $\zeta = 1/\sqrt{2}$, or $k =$

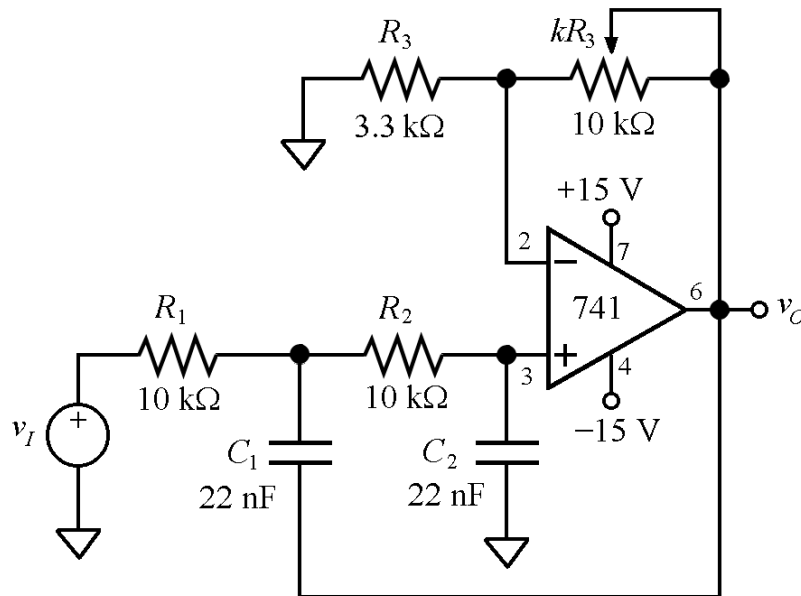


Fig. 12 – Circuit to investigate 2nd-order responses.

$2 - \sqrt{2}$. This will set up the circuit for what is known as the *maximally-flat*, or *Butterworth* frequency response. Then, using a procedure similar to that for the 1st-order circuit, measure the frequency response, and sketch and label the experimental magnitude Bode plot.

MC11: Repeat Step MC10, but for $\zeta = 0.2$, which corresponds to a much less damped response. You will also notice that in the frequency region near of ω_0 the response now exhibits a significant amount of *peaking*, an effect that is absent in the maximally-flat case of the previous step.

S12: Simulate the circuits of Steps M8 and M9 via PSpice, display their magnitude Bode plots, compare with the experimental ones of Steps MC10 and MC11, and account for possible differences between measurement and simulation.

M13: We are now going to observe the *time-domain* behavior of the circuit of Fig. 12. To this end, adjust the waveform generator so that v_I is a *square wave* alternating between -1 V and $+1$ V with an initial frequency of a few hundred Hz. Then slowly vary the wiper from the left to the right to see how the response changes from *non-oscillatory* ($k = 0$), to a *damped sinusoid* ($0 < k < 2$), to a *sustained sinusoid* ($k = 2$ exactly), to a *diverging sinusoid* ($k > 2$). Note that in the latter case the sinusoid will diverge until the saturation limits of the op amp are reached, thus preventing any further amplitude growth. Moreover, the circuit will continue to oscillate irrespective of v_I , something you can actually verify by letting $v_I \rightarrow 0$. Show your results to the lab instructor.

Record the waveform of v_O for *three* different values of k , namely, $k < 2$, $k = 2$, and $k > 2$. The first and last values of k are of your own choice, so make sure that once you are finished, you measure the corresponding values of kR_3 and R_3 in order to compute the actual values of k (pull them out of the circuit to measure them!) Comment on all of your results

S14: Simulate the circuit of Step M13 via PSpice for each of the three values of k . Hence, compare with Step M13, and account for possible differences.

C15: For *each* of the *three* values of k above: (a) develop a mathematical expression for $v_O(t)$ (for the

case $k > 2$, such an expression would hold only up to the onset of op amp saturation), and (b) sketch and label on graph paper the s -plane location of the corresponding pole pair.

Sinusoidal Oscillator:

Instability can be exploited on purpose to implement sine wave oscillators. With $k > 2$, the circuit of Step M13 is indeed an oscillator, but a rather crude one because of the abrupt clipping due to op amp saturation. Figure 13 shows an *automatic amplitude-control* mechanism that uses a diode network to ensure rounder edges for the sine wave, and, thus, much less distortion.

We can get the circuit to *start* oscillating on its own, without any externally applied input v_i , by letting $k > 2$. Indeed, just the noise present at the input of the op amp will suffice to initiate oscillations. As the oscillations build up, the diodes will finally start conducting (D_1 on the positive alternations, D_2 on the negative alternations), effectively placing R_4 in *parallel* with kR_3 and thus *decreasing* the effective value of k . You should be able to convince yourself that the circuit will stabilize itself at the amplitude that makes $k = 2$ *exactly*. We can visualize the process in the s -plane by saying that we are starting out with the pole pair in the *right-half* of the s -plane to ensure oscillation buildup, and then use *nonlinear amplitude control* to pull the pole pair back *right on the imaginary axis* to maintain sustained oscillation.

M16: With power off, assemble the circuit of Fig. 13 (note the absence of any input!) Apply power, and starting with the wiper all the way to the left, vary it gradually until oscillation just starts to build up, and then stabilizes at a sine wave. With the oscilloscope's time scale set to display about a couple of cycles, fiddle with the wiper until you empirically find the setting that makes your sine wave as *pure* as possible.

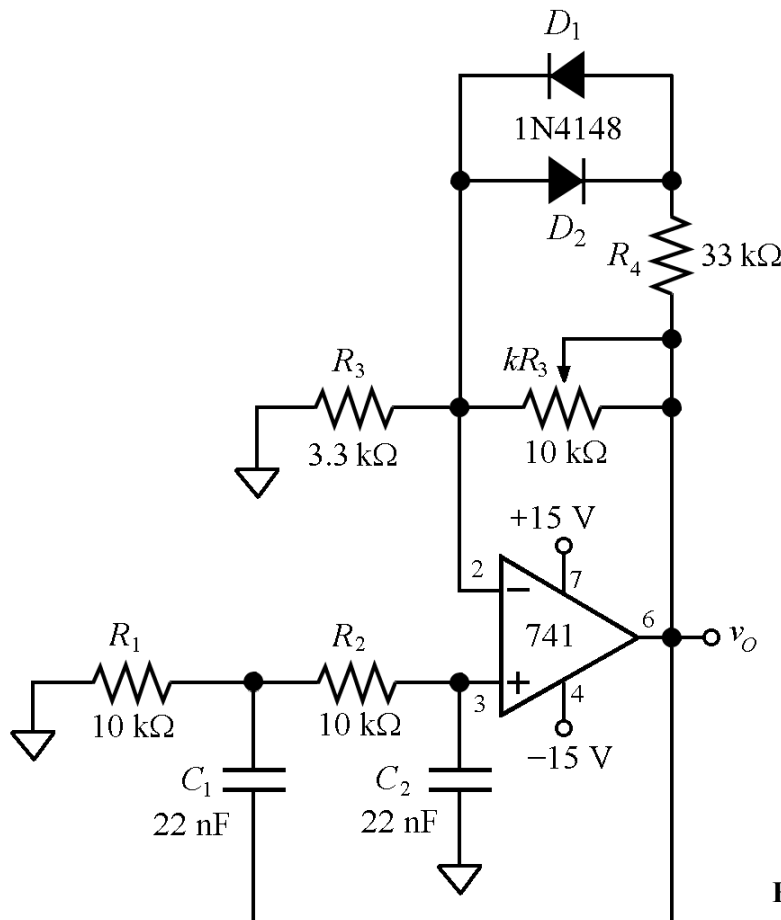


Fig. 13 – Sinusoidal oscillator.

What is its frequency f_0 ? How does it compare with the theoretical value $f_0 = 1/2\pi RC$, $R = R_1 = R_2$, $C = C_1 = C_2$?

S17: Simulate the circuit of Step M16 via PSpice, and display *two* versions of the output waveform: (a) a *coarse* one showing how it builds up, and (b) a *finer* one showing just a couple of cycles after the oscillation has stabilized. Compare the frequency with that measured in Step M16, and account for possible differences.