0.0.1 Euler's Method of Approximating Solutions to Differential Equations $y' = f(t, y)$

1. Ideally, the direction field provides *continuous* guidance as we trace the solution to an IVP. In practice we cannot accept or use continuous information. We can only use *discrete* bits of information.

   (a) **Recall:** if you have incomplete information about a function $y(t)$, but you know at $t = a$ the value $y(a)$ and $y'(a)$, then for a small value dt you can approximate $y(a + dt) \approx y(a) + y'(a)\, dt$

2. Let's go back to $y' = t^2 + y^2$, $y(0) = 1$.

   (a) If we start at $t = 0$, $y = 0$, the slope of the solution curve is 0. So we might go to $t = 0.1$. Then $y(0.1) \approx y(0) + y'(0)\, 0.1 = 0$. We can go to $(0.1, 0)$ and take another direction reading.

   (b) At the point $t = 0.1$, $y = 0$, the slope is $y'(0.1) \approx 0.1^2 + 0^2 = 0.01$. Then $y(0.2) \approx y(0.1) + y'(0.1)\, 0.1 \approx 0 + 0.01 \times 0.1 = 0.001$. We can go to $(0.2, 0.001)$ and take another direction reading.

   (c) Let's make a chart

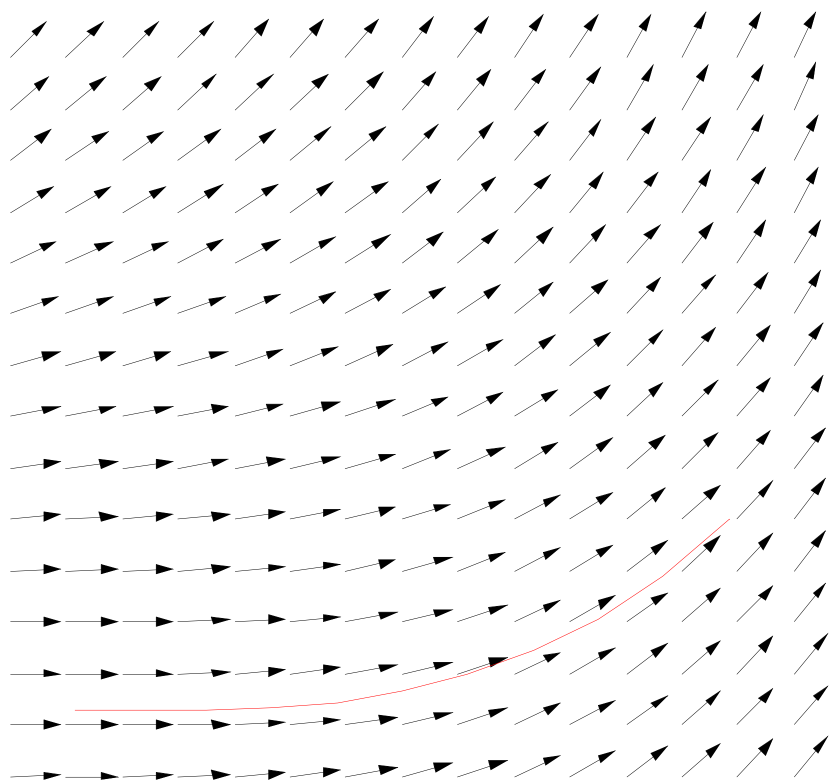
t	y
0.	0.
0.1	0.
0.2	0.001
0.3	0.0050001
0.4	0.0140026
0.5	0.0300222
0.6	0.0551123
0.7	0.0914161
0.8	0.141252
0.9	0.207247
1.	0.292542

   i. For those interested, I did this with some fairly fancy *Mathematica*. You can look up the commands in the *Mathematica* help file.

```
In[25]:= yp[t_, y_] := t^2 + y^2;
         t = 0.;
         y = 0.;
         n = 10;
         dt = 1/n;
         ylist = {{t, y}};
         Do[y = y + yp[t, y] dt; t = t + dt;
           AppendTo[ylist, {t, y}], {n}];
         Prepend[ylist, {"t", "y"}] // TableForm
```

   (d) We can plot our solution:, combined with the direction field

```
Needs["Graphics`Graphics`"];
Needs["Graphics`PlotField`"];
DisplayTogether[
 PlotVectorField[
  {1, tt^2 + yy^2} / Sqrt[1 + (tt^2 + yy^2)^2],
  {tt, -0.1, 1.1}, {yy, -0.1, 1}],
 ListPlot[ylist,
  PlotStyle → RGBColor[1, 0, 0],
  PlotJoined → True]]
```



3. Class try: $y' = t + y$, $y(-0.2) = 0$. Use $dt = 0.2$ and go to $t = 1$. Then graph solution.

0.0.2 Exponential Growth and Decay

1. Let $P(t)$ represent the size of some population (of rabbits, bacteria, radioactive isotopes, dollars, etc) at time t.
   (a) In an application the units will be specified: P might be measured in individuals or millions of individuals or moles or thousands of dollars, etc.; t might be measured in microseconds or seconds or days or years, etc.
   (b) For the general theory the choice of units is not important
2. An assumption often made in this situation is that the rate of growth of P is proportional to the size of P :
$$\frac{dP}{dt} = kP$$
   (a) Of course, this means that $P(t) = Ce^{kp}$ for some constant C
   (b) The values of k and C can be determined by observing the population at two separate times.

(c) Suppose $P(0) = 10,000$ and P is growing 10% each time period.

i. Then $P(1) = 11,000$ and we have

$$\begin{aligned} 10000 &= Ce^{k0} \\ 11000 &= Ce^{k} \\ C &= 10000 \\ k &= \ln(1.1) \\ &= 0.09531 \end{aligned}$$

(d) k is called the *instantaneous growth rate*

3. Class: a population of bacteria starts out with 5,783,000 cells. It gows exponentially, and after 3 hours there are 10,375,000 cells.

(a) What units would you use for time in this situation

(b) what units would you use for population

(c) What is the population equation for the bacteria? Find the constants.

(d) How many bacteria will we have after 5 hours?

4. Class: theoretical question. If a population satisfies the differential equation $P' = kP$, how long (as a function of k) does it take for the population to double? Do you need to know the initial population level?

5. Radioactive decay (is the decay radioactive or is the radiation decaying?)

(a) radioactive decay is an example of a process where a population is decreasing rather than increasing at a steady rate.

(b) It satisfies the same differential equation as exponential growth: $P' = kP$, but $k < 0$.

(c) The *half-life* of a radioactive material is the time it takes for half of it to decay

(d) Example: a certain radioactive material has a half-life of 1534 years. If you initially have 17.3 mg of radioactive material, how much will there be after 2000 years?

i. two solutions. Easy one: $P(t) = 17.3 \times 2^{-t/1534}$ so $P(2000) = 17.3 \times 2^{-2000/1534} = 7.0076$ mg

ii. hard one: $P(t) = Ce^{kt}$ where $P(0) = 17.3$ and $P(1534) = \dfrac{17.3}{2}$. Thus $C = 17.3$ and $k = \dfrac{-\ln 2}{1534}$. Thus $P(2000) = 17.3e^{\frac{-\ln 2 \times 2000}{1534}} = 7.0076$ mg

(e) Class: a radioactive material decays from 5.3 mg to 4.6 mg in 18 seconds. Find the half-life.

6. If P_0 is invested at 6% annual interest rate, the amount after time t is

$$P = P_0 (1.06)^t$$

7. Class: If a bank account pays 5% semiannually in quarterly payments, what percentage does it pay each quarter?

8. You can write the interest formula as an exponential formula

(a) For example, 6% annual interest becomes

$$\begin{aligned} P &= P_0 e^{t \ln(1.06)} \\ &= P_0 e^{0.0583t} \end{aligned}$$

but why? There might be some theoretical interest in this, particularly if the interest rate was changing, as it would be in a more complicated model of financial growth.

(b) If the instantaneous rate of return is a function $r(t)$ of time (like instantaneous speed), and if an investment is worth P_0 at time $t = 0$, then we can calculate the worth $P(t)$ of the investment at time t.

$$\begin{aligned} P'(t) &= P(t)\, r(t) \\ P(0) &= P_0 \end{aligned}$$

i. This is a separable differential equation:

$$\begin{aligned} \int_0^t \frac{dP}{P} &= \int_0^t r(\tau)\, d\tau \\ \ln(P(t)) - \ln(P(0)) &= \int_0^t r(t)\, dt \\ P(t) &= P_0 e^{\int_0^t r(t) dt} \end{aligned}$$

(c) Class do: if $r(t) = 0.05 \sin\left(\frac{\pi t}{180}\right)$, what is balance after 360 days. Suppose $r(t) = 0.05 \cos\left(\frac{\pi t}{180}\right)$

(d) Class: how to model a constant stream of payments.

$$\begin{aligned} P' &= P(1.05) + 5000 \\ P(0) &= 0 \end{aligned}$$

How much after 1 year? After 5 years?