SIGNIFICANT FIGURES

Significant figures are those digits in a number that are known with certainty plus the first digit that is uncertain.

Addition or subtraction: In addition or subtraction of terms, the number of significant figures in the result depends upon the first decimal position (left or right of the decimal) where uncertainties in any one of the terms arise. Consider the sum shown below.

Adding the numbers as they are, one gets the result shown. But there is uncertainty in the second and third decimal places to the right of the decimal. (81.6 could perhaps be 81.62 or 81.563.) So, considering significant figures, the sum would be expressed as 103.8. Notice that the result has four significant figures, even though none of the terms added has four significant figures.

$$\begin{array}{r} 3.74 \\ 81.6 \\ 16.559 \\ 1.7 \\ 0.195 \\ \hline 103.794 \end{array} \quad (103.8)$$

Multiplication and division: In multiplication and division (and also for powers and roots), the result of the mathematical operation has no more significant figures than the term which has the least number of significant figures. For example:

$$\frac{5.1 \times 0.00573}{0.1744} = 0.17$$

Only two figures are significant in the result, since one of the terms involved has only two significant figures.

An exception: There are exceptions to these rules which arise from time to time.

Suppose, for example, you are instructed to make five independent determinations of some quantity and then find the average value. If your five measurements produce the values 5.73, 5.68, 5.66, 5.71, and 5.68, the sum of these is 28.46 as shown at the right. In this case, the addition of five terms, each with three significant figures, yields a sum with four significant figures.

$$\begin{array}{r} 5.73 \\ 5.68 \\ 5.66 \\ 5.71 \\ 5.68 \\ \hline 28.46 \end{array}$$

When the sum is divided by five to determine the average value, the result is:

$$\frac{28.46}{5} = 5.692 \quad (5.69)$$

Although the "5" in the denominator is written as having only one significant figure, it is *exactly* five, so the number of significant figures to be retained in the result depends on the numerator. The numerator has four significant figures, and using the rules stated above, four significant figures should appear in the result. However, an exception is made in this case. Since the five readings were recorded to only three significant figures, the average value is good to only three significant figures. The average is therefore expressed as 5.69. A calculation of the possible error in the average would show that only three significant figures are valid.

Another Exception: There are other cases where a calculation of possible error indicates that you should keep one more or one less figure than you would if you followed the preceding rules. Suppose you have just calculated a quantity Q, and the answer displayed on your calculator is 165.34329 (eight digits!). Using the rules for significant figures, you determine that only three of the figures are significant, so you round off the result to Q = 165. You then calculate the possible error in Q, you find it to be $\Delta Q = \pm 0.7$. Since the decimal location of the possible error is in the first place to the right of the decimal, this indicates that one more figure in the result has some significance and should be kept. That is, the quantity expressed with its error should be Q = 165.3 ± 0.7.

Zeros: Zeros between two non-zero numbers are significant figures. The number 101.07 contains five significant figures. Zeros used to locate a decimal are *not* significant figures. For example, 0.00015 contains two significant figures. The number 17,300 (where the final two zeros are merely "spacers" to locate the decimal space) contains three significant figures. If these numbers were expressed in scientific notation they would be 1.5×10^{-4} and 1.73×10^{4}, with two and three significant figures respectively. A zero which is *not* between two other non-zero numbers can sometimes be a significant figure. For example 1.50×10^{-4} contains three significant figures and would be written in decimal form as 0.000150. Note that a zero on the end of a number to the right of a decimal *is* a significant figure. Similarly, 1.730×10^{4} contains four significant figures, and would be written in decimal form as 17,3$\underline{0}$0. Note that the zero is underlined to indicate that it *is* a significant number.

Logarithms: When taking logarithms of numbers, the *mantissa* of the logarithm should have the same number of significant figures as the number whose log is being taken. The *characteristic* is an indicator of the exponent when the number is expressed as a power of ten, and it will provide an additional significant figure in the logarithm of the number. This means that, except for numbers between one and ten, the logarithm of a number will have one more digit than the significant figures in the number itself.

For example, expressing the following logarithms of numbers to the appropriate number of significant figures, we have:

$$\log 2.53 = \log (2.53 \times 10^{0}) = 0.403$$
$$\log 25.3 = \log (2.53 \times 10^{1}) = 1.403$$
$$\log 253 = \log (2.53 \times 10^{2}) = 2.403$$
$$\log 253.2 = \log (2.532 \times 10^{2}) = 2.4035$$

Note that in these examples, the first three numbers vary by only a decimal place, so the mantissa of the logarithm of each is the same (.403), and it is this mantissa which is expressed to the same number of significant figures as the number. Since the number 253.2 has four significant figures, the mantissa of its logarithm is expressed to four significant figures. Since the characteristec of the last three logarithms is non-zero, the logarithms have one more figure than the significant figures in the number itself.

Calculating Possible and Probable Errors

Consider the function V(x,y,z). If a calculated value V is to be found using values x, y and z with possible errors Δx, Δy, Δz, then the possible error is given by

$$\Delta V = \left|\frac{\partial V}{\partial x}\right|\Delta x + \left|\frac{\partial V}{\partial y}\right|\Delta y + \left|\frac{\partial V}{\partial z}\right|\Delta z .$$

So if $V(x,y,z) = x + y + z$

$$\Delta V = \Delta x + \Delta y + \Delta z .$$

Or if $V(a,b,c,d) = 2a - 3b + 4c - 2d$

$$\Delta V = 2\Delta a + |3|\Delta b + 4\Delta c + |2|\Delta d .$$

Notice that the absolute value of the derivatives must be used.

If $V(x,y) = yx$,

$$\Delta V = y\Delta x + x\Delta y.$$

If $V(x,y,z) = xy - \frac{1}{2} y^2 z$,

$$\Delta V = y\Delta x + |x - yz|\Delta y + \frac{1}{2} y^2 \Delta z .$$

Again, the absolute value of the partial derivatives is used.

If the probable error in a calculated value V(x,y,z) is desired, it can be shown that the probable error δV is given by

$$\delta V = \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{\partial V}{\partial y}\right)^2 (\delta y)^2 + \left(\frac{\partial V}{\partial z}\right)^2 (\delta z)^2} .$$

If x, y and z are measured values, then δx, δy and δz represent the estimated error in the measurements (i.e. the same as Δx, Δy, Δz). However, if x had been function x(a,b), δx would be a calculated *probable* error.

If V is a sum of numbers, then

$$V(x,y,z) = x + y + z ,$$

since

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 1 ,$$

$$\delta V = \sqrt{1^2 \cdot (\delta x)^2 + 1^2 \cdot (\delta y)^2 + 1^2 \cdot (\delta z)^2}$$

$$\delta V = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2} .$$

If V is a product of two numbers , then

$$V(x, y) = xy$$

$$\frac{\partial V}{\partial x} = y \qquad \frac{\partial V}{\partial y} = x$$

$$\delta V = \sqrt{y^2 (\delta x)^2 + x^2 (\delta y)^2} .$$

As a numerical example, let us find the density of a sphere as well as the possible and probable error involved. Assume that the mass m = (83.1 ± 0.1) g and the diameter d = (2.55 ± 0.02) cm. Since density (ρ) is mass per unit volume, the density is

$$\rho = \frac{m}{V} = \frac{m}{\left(\frac{4}{3}\right)\pi r^3} = \frac{m}{\left(\frac{4}{3}\right)\pi\left(\frac{d}{2}\right)^3}$$

$$\rho = \frac{6m}{\pi d^3}$$

$$\rho = \frac{(6)(83.1\,\text{g})}{\pi(2.55\,\text{cm})^3} = 9.57\frac{\text{g}}{\text{cm}^3} .$$

The possible error $\Delta\rho$ is given by

$$\Delta\rho = \left|\frac{\partial\rho}{\partial m}\right|\Delta m + \left|\frac{\partial\rho}{\partial d}\right|\Delta d$$

$$\Delta\rho = \frac{6}{\pi d^3}\,\Delta m + \frac{18m}{\pi d^4}\,\Delta d$$

$$\Delta\rho = \frac{(6)(0.1\,\text{g})}{\pi(2.55\,\text{cm})^3} + \frac{(18)(83.1\,\text{g})(0.02\,\text{cm})}{\pi(2.55\,\text{cm})^4}$$

$$\Delta\rho = 0.01\frac{\text{g}}{\text{cm}^3} + 0.2\frac{\text{g}}{\text{cm}^3}$$

$$\Delta\rho = 0.2\frac{\text{g}}{\text{cm}^3} .$$

Expressed with its possible error, $\rho = (9.6 \pm 0.2)\,\frac{\text{g}}{\text{cm}^3}$.

Notice that when one observes the rules for handling significant figures in calculating the possible error, the error term involving the diameter is the only one affecting the final result. The reason for this is that the mass is known to 1 part in 831, whereas the diameter is known only to about 1 part in 128. Also, the diameter term is cubed, which means that its error of 1 part in 128 is tripled, yielding a final error of 3 parts in 128. This is an error of 1 part in 43, or 0.2 g/cm^3 in the calculated density of 9.6 g/cm^3. Since the error calculation indicates uncertainty in the first place to the right of the decimal, the density is expressed with only two significant figures rather than the three previously determined. This is an example of how exceptions to the rules for handling significant figures must be made when possible errors are known. This particular case has resulted in one less significant figure than one might otherwise have thought justified, but there are also situations in which an additional figure is found to be significant.

The probable error $\delta\rho$ is given by

$$\delta\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\right)^2 (\delta m)^2 + \left(\frac{\partial\rho}{\partial d}\right)^2 (\delta d)^2}$$

$$\delta\rho = \sqrt{\left(\frac{6}{\pi d^3}\right)^2 (\delta m)^2 + \left(\frac{18m}{\pi d^4}\right)^2 (\delta d)^2}$$

$$\delta\rho = \sqrt{\left(\frac{(6)(0.1\text{ g})}{\pi(2.55\text{ cm})^3}\right)^2 + \left(\frac{(18)(83.1\text{ g})(0.02\text{ cm})}{\pi(2.55\text{ cm})^4}\right)^2}$$

$$\delta\rho = \sqrt{0.0001\frac{g^2}{cm^6} + 0.05\frac{g^2}{cm^6}}$$

$$\delta\rho = \sqrt{0.05\frac{g^2}{cm^6}}$$

$$\delta\rho = 0.2\frac{g}{cm^3}\ .$$

In this case, the probable error is equal to the possible error.

In some cases the experimental value of a quantity used in the calculation of another quantity will be the result of a single measurement, and the estimated error in that measurement will be used in error calculations. In other cases the experimental value used in a calculation will result from a number of measurements which have been averaged. In the latter, the estimated error in the value used in error calculations may also be a result of the numerous measurements made, and it may be in the form of a mean deviation or root-mean-square deviation.

If n measurements are made of a value (call them y_1, y_2, y_3, ... y_n), the average or mean value is equal to

$$\bar{y} = \sum_{i=1}^{n} \frac{y_i}{n} .$$

The average deviation or mean deviation is simply the average absolute value of the deviation of each measurement from the mean. Expressed algebraically,

$$\text{mean deviation} = \left(\frac{1}{n}\right)\sum_{i=1}^{n} |y_i - \bar{y}| .$$

The root-mean-square deviation is the square root of the average of the squares of the deviations from the mean. Expressed algebraically,

$$\text{r.m.s deviation} = \sqrt{\left(\frac{1}{n}\right)\sum_{i=1}^{n} |y_i - \bar{y}|^2} .$$

Neither the mean deviation nor the r.m.s. deviation is the same as the standard deviation of statistical analysis. Values for standard deviations computed using built-in programs in pocket calculators should not be used when either the mean deviation or the r.m.s. deviation are requested in laboratory reports. For large numbers of measurements (greater than 10, keeping one significant figure) the r.m.s. deviation and the standard deviation will be the same.