NOTE: Use of calculators will NOT be permitted. You have 50 minutes to complete this test.

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BONUS QUESTIONS

1. (1pt) What is the exact value of the angle between two carbon-to-carbon bonds in the molecular structure of diamond?

2. (1pt) What is the value of \( \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} \) if the variables \( x, y, z \) satisfy a differentiable relation?

3. (1pt) Let \( \varepsilon = \varepsilon(a,b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2 \) be the sum of least squares error for a random data sample of points \( (x_i, y_i), i = 1, \ldots, n \). What is the sign of \( \varepsilon_{aa} \varepsilon_{bb} - \varepsilon_{ab}^2 \) and that of \( \varepsilon_{aa} \) ?

4. (1pt) What is the value of
\[
\begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix}
\]
where \( x = r \cos \theta, y = r \sin \theta \).

5. (1pt) What is the value of
\[
\begin{vmatrix}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}
\end{vmatrix}
\]
where \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \).
Part 1. (30pts - 5 questions)

1. Find the average of the function \( f(x, y, z) = xyz \) along the line segment from \((1, 0, 1)\) to \((2, 2, 3)\).
   
   (A) \( \frac{2}{3} \)  
   (B) \( \frac{4}{3} \)  
   (C) \( \frac{2}{5} \)  
   (D) \( \frac{4}{5} \)  
   (E) None of the above

2. Find the circulation \( \int_C \mathbf{F} \cdot d\mathbf{r} \) of the vector field \( \mathbf{F} = 2xz(\ln y)i + \frac{x^2z}{y}j + x^2(\ln y)k \) along the path \( C \) from \((1, 1, 1)\) to \((2, 2, 2)\) formed by two line segments meeting at the point \((2, 3, 1)\).

   (A) \( 8 \ln 3 \)  
   (B) \( 9 \ln 3 \)  
   (C) \( 8 \ln 2 \)  
   (D) \( 9 \ln 2 \)  
   (E) None of the above

3. Find the outward flux \( \oiint_S \mathbf{F} \cdot d\mathbf{S} \) of the vector field \( \mathbf{F} = x^2i + xzj + 3zk \) across the surface of the cube bounded by the planes \( x = 1, y = 1, z = 1 \) and the coordinate planes.

   (A) \( 2 \)  
   (B) \( 4 \)  
   (C) \( 6 \)  
   (D) \( 8 \)  
   (E) None of the above

4. Find the outward flux \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \) of the vector field \( \mathbf{F} = xi + yj \) across the circle \( C \) of radius one centered at the origin.

   (A) \( 2\pi \)  
   (B) \( 4\pi \)  
   (C) \( 6\pi \)  
   (D) \( 0 \)  
   (E) None of the above
5. Find the flux of the curl \( \int_S (\nabla \times \mathbf{F}) \cdot d\sigma \) of the vector \( \mathbf{F} = -yi + xj \) across the paraboloid \( z = 4 - x^2 - y^2 \) with the upward orientation.

(A) \( 2\pi \)
(B) \( 4\pi \)
(C) \( 6\pi \)
(D) \( 8\pi \)
(E) None of the above

Part 2. (10pts)

(a) (6pts) Let \( f(x, y, z) = x^3 + yz^2 \) and \( \mathbf{F}(x, y, z) = x^2zi + yzj + y^2k \). Compute

\[
\nabla f =
\]

\[
\nabla \cdot \mathbf{F} =
\]

\[
\nabla \times \mathbf{F} =
\]

\[
\nabla \times (\nabla f) =
\]

\[
\nabla \cdot (\nabla \times \mathbf{F}) =
\]

\[
\nabla \cdot (\nabla f) =
\]

(b) (4pts) Let \( \mathbf{r} = \frac{x\mathbf{i}}{\sqrt{x^2 + y^2}} + \frac{y\mathbf{j}}{\sqrt{x^2 + y^2}} \) and \( \mathbf{\theta} = \frac{-yi}{\sqrt{x^2 + y^2}} + \frac{xj}{\sqrt{x^2 + y^2}} \).

IDENTIFY the true statement(s) among the following.

(A) \( \mathbf{r} \cdot \mathbf{\theta} = 0 \) \hspace{1cm} (B) \( \nabla \cdot \mathbf{r} = 0 \) \hspace{1cm} (C) \( \nabla \cdot \mathbf{\theta} = 0 \) \hspace{1cm} (D) \( \nabla \times \mathbf{r} = 0 \) \hspace{1cm} (E) \( \nabla \times \mathbf{\theta} = 0 \)
Part 3. (10pts)

(i) Explain why the circulation of a conservative vector field along a closed curve vanishes.

(ii) Give an example of an irrotational vector field that is not conservative.

(iii) Explain how you know your answer to (b) is irrotational and not conservative.

(iv) Under what additional condition will an irrotational vector field be conservative?

(v) What kind of vector field has the property that its flux across any closed surface vanishes?