Isomorphism in Digital Elevation Models and Its Implication to Interpolation Functions

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Abstract
Terrain is an ordered surface where locations relate to each other through their elevations. Preservation of this topographic orderliness by an interpolation method so that “if point A is higher than point B in the terrain, the interpolated elevation of A remains higher” is important to assure the practical value of the resultant DEM. Based on the concepts in group theory, this paper points out that a DEM must be isomorphic to terrain surface in order to preserve the topographic orderliness. Such a DEM, if generated through interpolation, can only be obtained if the interpolation function is an isomorphism. Two necessary conditions for isomorphism are identified: a one-to-one relationship between the topographic surface and the surface corresponding to the interpolation function, and the feasibility to configure the topographic surface into monotonic patches during interpolation. The isomorphism of three widely used interpolation methods is examined. It is found that linear interpolation in 1D and TIN are both isomorphisms, meaning that a DEM interpolated by either method can preserve the topographic orderliness. In contrast, bilinear interpolation is not an isomorphism. Considering the practical challenges in assuring the necessary conditions of isomorphism in each method, linear interpolation in 1D is recommended as the optimal method to interpolate a DEM.

Introduction
Terrain surface is made up of innumerous points, each of which has a location and a determined, though possibly unknown, elevation. These points are neither randomly distributed nor independent of each other. Rather, they are organized according to their elevations to result in terrain features such as hilltops, valleys, saddles, etc. From this perspective, it can be said that terrain has an embedded orderliness. The topographic orderliness of terrain is best demonstrated by the drainage systems where surface water flows systematically into tributaries and stream channels until it reaches the outlets eventually. Because topographic orderliness is an inherent characteristic of terrain, any effort to assure it can be said that terrain has an embedded orderliness. Such a DEM, if generated through interpolation, can only be obtained if the interpolation function is an isomorphism. Two necessary conditions for isomorphism are identified: a one-to-one relationship between the topographic surface and the surface corresponding to the interpolation function, and the feasibility to configure the topographic surface into monotonic patches during interpolation. The isomorphism of three widely used interpolation methods is examined. It is found that linear interpolation in 1D and TIN are both isomorphisms, meaning that a DEM interpolated by either method can preserve the topographic orderliness. In contrast, bilinear interpolation is not an isomorphism. Considering the practical challenges in assuring the necessary conditions of isomorphism in each method, linear interpolation in 1D is recommended as the optimal method to interpolate a DEM.

Isomorphism and Homomorphism in DEM Generation
Let $a$ and $b$ be two points whose true elevations are $z_a$ and $z_b$, respectively. Their elevations in an interpolation-generated DEM $z_a^*$ and $z_b^*$ are obtained through interpolation. This type of DEM usually uses the contour lines and spot elevations in a topographic map as the source data, e.g., USGS Level 2 and Level 3 DEMs (USGS, 1998). It can also be interpolated from lidar point data or another grid DEM of coarse resolution. Interpolation from a coarse-resolution DEM is often referred to as DEM resampling, i.e., to re-interpolate a DEM to higher resolution (Rees, 2000; Shi et al., 2005). In the following discussion, we shall use contour lines as the source data. However, the derivation process can be easily extended to DEMs interpolated from points.

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DEMs are $Z_a$ and $Z_b$, respectively. The ability of a DEM to preserve topographic orderliness can be expressed as follows: If $z_a < z_b$, will $Z_a < Z_b$ and vice versa? To answer this question, the relationship between topographic surface, DEM, and interpolation function must be clarified first. On this, we resort to the concept of homomorphism and isomorphism in abstract algebra to point out that (a) a DEM preserving topographic orderliness must be homomorphic to the topographic surface, and (b) a DEM preserving topographic orderliness can only be obtained if the interpolation function is an isomorphism.

**Homomorphism and Isomorphism Defined**

The word homomorphism is from Greek language and literally means “similar shape.” It refers to the correspondence between two sets which may contain entirely different elements but actually similar in terms of form, appearance, or size. Mathematically, homomorphism is defined as follows:

**Definition 1**

Let $f$ be a function to map set $A$ to set $A'$. If $a < b$ entails $f(a) < f(b)$ for any two elements $a$ and $b$ in set $A$, then $A$ and $A'$ are homomorphic and $f$ is called a homomorphism.

In the context of DEM generation, terrain surface is one algebraic set $A = \{ (x,y,z) \}$, DEM is another algebraic set $A' = \{ (x,y), Z \}$, interpolation function is $f$ which maps terrain surface to a DEM, i.e., $f: A \rightarrow A'$. To facilitate discussion, we shall use $z$ and $Z$ to refer to the true and interpolated elevation throughout the paper. For example, $z_a$ and $Z_a$ are the true and estimated elevation of point $a$, respectively. If a DEM preserves topographic orderliness, there must be $z_a < z_b$ entails $Z_a < Z_b$, for any two points $a$ and $b$. Per Definition 1, such a DEM and the topographic surface must be homomorphic. Furthermore, the interpolation function utilized is a homomorphism.

A homomorphic interpolation function assures that if point $a$ is higher than $b$ in the actual terrain, the interpolated elevation of point $a$ in a DEM will remain higher. From the practical application perspective, the vice versa should also be actualized, i.e., if a DEM suggests that point $a$ is higher than point $b$, a should be indeed higher in the actual terrain. Only in this way, the topographic orderliness in a DEM will agree with the ground truth. To this end, we introduce the concept of bijection or bijective function:

**Definition 2**

A function $f: A \rightarrow A'$ is bijective if and only if for every element $a'$ in $A'$ there is exactly one element $a$ in $A$ such that $f(a) = a'$. In other words, there is a one-to-one correspondence between $A$ and $A'$.

**Definition 3**

$f: A \rightarrow A'$ is an isomorphism if it is a homomorphism and a bijection function. Formally, the definition of isomorphism also requires $f^{-1}: A' \rightarrow A$ to be a homomorphism. However, this requirement is automatically satisfied as long as $f^{-1}$ exists and $f$ is a bijective homomorphism. This can be proven through refutation. Suppose $f^{-1}: A' \rightarrow A$ is not a homomorphism, meaning that $f(a) > f(b)$ does not necessarily entail $a > b$, hence there exists the possibility of $a \leq b$. However, if $a \leq b$, there must be $f(a) \leq f(b)$ because $f$ is a bijective homomorphism. This contradicts the given condition of $f(a) > f(b)$, suggesting that it is impossible for $f^{-1}: A' \rightarrow A$ to be not a homomorphism. Since in DEM interpolation $f^{-1}$ always exists because it is the function to map a DEM point to a terrain point, isomorphism is equivalent to bijective homomorphism.

We now present the argument that an interpolation method must be an isomorphism in order to assure that the resultant DEM preserves topographic orderliness. To reach this conclusion, we first prove that a valid DEM interpolation method must be bijective, i.e., a one-to-one correspondence exists between terrain surface and the surface created by the interpolation function.

**A Valid Interpolation Function must be Bijective**

A grid DEM provides elevation for a set of selected locations only. However, every point on the topographic surface has an opportunity to be included in a DEM because the spatial resolution of a DEM can be infinitely small in theory. This suggests that a valid interpolation function $f$ must assure that an estimated elevation $f(x, y)$ exists at any location $(x, y)$. From this perspective, $f$ is a continuous function. Let $A' = \{ (x,y,f(x,y)) \}$. Since $A'$ is de facto a surface, it will be referred to as the interpolation surface in subsequent discussions. It can be seen that a grid DEM is essentially a subset of $A'$.

Given a location $(x, y)$ on the topographic surface, its true elevation $z$ is a single, determined, though possibly unknown value (in the case of vertical cliffs and overhangs, only the highest elevation point belongs to the topographic surface). Since each $(x,y)$ corresponds to exactly one $z$, the estimated elevation $f(x,y)$ must also be a single, determined value; otherwise there is no means to quantify the interpolation error which is defined as $z - f(x,y)$. This suggests that every point $(x, y, z)$ in the topographic surface must correspond to exactly one point $(x, y, f(x,y))$ in the interpolation surface, and vice versa. Per Definition 2, interpolation function $f$ must be bijective.

That an interpolation function must be bijective is a necessary condition to interpolate a DEM, regardless whether the resultant DEM preserves topographic orderliness. Most interpolation functions can guarantee this one-to-one correspondence between the topographic surface and the interpolation surface at most locations; the challenge is whether such relationship exists everywhere. Many interpolation functions are patchwise polynomials which divide the terrain into contiguous patches so that interpolation can be conducted patch by patch. An example is Triangulated Irregular Network (TIN) interpolation which approximates terrain surface using triangle patches. For points located on the boundary of two triangle patches, its elevation can be estimated by either patch. The one-to-one correspondence between the topographic surface and the interpolation surface requires that the two estimated values must be exactly the same. Otherwise multiple estimates exist at a single location hence creates ambiguity to the DEM users. Post-processing such as averaging the multiple estimates can enforce the interpolated elevation at each location to be a single value. However, such processing is rather ad hoc because of the lack of theoretical justification. It is much desired if an interpolation method can assure the one-to-one relationship automatically and proactively. In the next section, we will examine the ability of three interpolation methods to assure bijection.

**An Interpolation Function must be an Isomorphism to Preserve Topographic Orderliness**

In the previous sections, we obtained that a valid DEM interpolation method $f: (x,y,z) \rightarrow (x,y,f(x,y))$ must be bijective. On the other hand, an interpolation function must be a homomorphism in order to assure that high-elevation points in the topographic surface remain high
in the resultant DEM. According to Definition 3, such an interpolation function which is a bijective homomorphism must be an isomorphism. In other words, an interpolation function must be an isomorphism in order to result in a DEM preserving topographic orderliness.

Isomorphism provides a completely new angle to evaluate an interpolation function. In the rest of the paper, we examine three interpolation methods, namely linear interpolation in 1D, TIN interpolation, and bilinear interpolation, to determine whether they are isomorphisms. These methods are widely employed in DEM interpolation and resampling. Their accuracy at DEM point scale, which is often measured by error variance and RMSE, has been studied intensively in the literature (Tempfli, 1980; Kidner, 2003; Zhu et al., 2005; Aguilar et al., 2006). Our research on isomorphism complements these studies by providing an insight into the ability of these methods to retain the interrelationship between DEM points. Unlike previous research which has been mainly conducted based on case studies or computer simulation, the research in this paper is based on mathematical proofs. The conclusions obtained can therefore be applied to any DEM generated by the interpolation function. In the following discussion, we introduce the three interpolation methods first, then examine whether all of them are bijective homomorphisms.

The Three Bijective Interpolation Methods

Linear interpolation in 1D, TIN, and bilinear interpolation are patchwise polynomial interpolation methods. When utilizing them to generate a DEM, the terrain is divided into patches; interpolation of the points in a patch is based on the reference data on that patch only. Because the ability to preserve topographic orderliness is an intrinsic property of an interpolation function, this paper will assume that the reference data are error free. In this way, the only error at a DEM point is due to the interpolation function itself.

Linear Interpolation in 1D

Linear interpolation in 1D is the method that human uses to visually estimate the elevation of a point on a topographic map. Let T be a point between two contour lines. The path by which surface water flows through T can be constructed (Figure 1). Linear interpolation in 1D approximates this flow path as a straight line ab. Since surface water always flows down gradient because of gravity or gravity-induced pressure, a flow path is monotonic in the sense that elevation decreases continuously along the path. The elevation of T is estimated as:

\[ Z_T = \omega z_a + (1 - \omega) z_b, \omega \geq 0 \]  

(1)

where \( \omega \) is the distance proportion of segment Th, i.e., \( \omega = \frac{Tb}{ab} \), \( z_a \) and \( z_b \) are error-free reference elevation at point a and b respectively, i.e., their true elevations. Since the solution to \( \omega \) is unique, there must be a one-to-one correspondence between a flow path and its straight line approximation (Figure 1).

Previously, we mentioned that the challenge to a patchwise interpolation method to be bijective is whether the one-to-one correspondence holds for points at the boundary of two patches. In the case of linear interpolation in 1D, the boundary of two patches is the shared endpoint of two flow paths, e.g., a in Figure 1. If a DEM point T is located at a, there is \( Z_T = z_a \) per Equation 1. Linear interpolation in 1D thus guarantees that every \((x, y)\) in the terrain has exactly one interpolated elevation, i.e., each \((x, y, z)\) corresponds to exactly one \((x, y, Z)\). Linear interpolation in 1D is therefore a bijection.

TIN Interpolation

TIN interpolation models the topographic surface as a set of triangle facets. The projection of each triangle facet on the horizontal plane is the triangle on which interpolation is conducted. For a terrain point \( T \), its interpolated elevation by TIN is:

\[ Z_T = \omega_a z_a + \omega_b z_b + \omega_c z_c, \omega_a + \omega_b + \omega_c = 1, \omega_a \omega_b \omega_c \geq 0 \]  

(2)

where \( z_a, z_b, \) and \( z_c \) are the elevations of the triangle vertices, and \( \omega_a, \omega_b, \omega_c \) are the areal proportions of the sub-triangles constructed using T (Figure 2). Given T, the values of \( \omega_a, \omega_b, \omega_c \) are uniquely determined, therefore the value of \( Z_T \) is single. In the case that a point falls on a triangle edge, e.g., p in Figure 2a, its interpolation by TIN is equivalent to linear interpolation on line \( bc \). This determined that the interpolated elevation of a point on the boundary of two triangle facets remains the same regardless of the triangle facet on which interpolation is conducted. Since each \((x, y, z)\) corresponds to exactly one \((x, y, Z)\) and vice versa, TIN is a bijection.

Bilinear Interpolation

Bilinear interpolation models terrain using quadrilateral patches. If each quadrilateral is a rectangle, bilinear interpolation can be written as:

\[ Z_T = \omega_a z_a + \omega_b z_b + \omega_c z_c + \omega_d z_d, \omega_a + \omega_b + \omega_c + \omega_d = 1 \]

\[ \omega_a \omega_b \omega_c \omega_d \geq 0 \]  

(3)

where \( \omega \) is the distance proportion of segment Th, i.e., \( \omega = \frac{Tb}{ab} \), \( z_a \) and \( z_b \) are error-free reference elevation at point a and b respectively, i.e., their true elevations. Since the solution to \( \omega \) is unique, there must be a one-to-one correspondence between a flow path and its straight line approximation (Figure 1).

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(3)
always coincide. Since a location (the elevation of a boundary point, but the estimates may not exactly the same. Both functions are equally valid to estimate the bicubic functions for two contiguous patches may not be points on that patch only. This creates the possibility that 14 coefficients involved are estimated based on the reference ration, each patch is modeled as a bicubic surface and the guaranteed bijective. In the case of the above bicubic interpo-

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tions. In the literature, higher-order patchwise polynomials

Monotonic Terrain Patches
To examine whether topographic orderliness is preserved in the three interpolation functions, we first examine whether these methods can divide terrain into a set of monotonic patches which splices with each other along terrain structure lines (e.g., ridges and valleys) or where slope gradients reverse. In other words, no “bumps” or “pits” should exist in a monotonic patch. This is because all three functions model a terrain patch as a flat facet, e.g., a line, a triangle, or a rectangle. If a terrain feature (e.g., a ridge, a peak, or a pit) falls inside a patch, it will be “smoothed out” hence lost in the interpolated surface. Consequently, topographic orderliness cannot be preserved.

Monotonic patches are the elementary units to conduct piecewise interpolation. Their existence is a necessary but not sufficient condition to preserve topographic orderliness. Both linear interpolation in 1D and TIN can satisfy this necessary condition. Linear interpolation in 1D is conducted on surface flow paths which are always monotonic because of gravity. In TIN, the edge or the vertex of a triangle can always be placed on a terrain feature such as peaks and pits. In the case that the initial triangulation does not result in monotonic patches, a triangle can always be further divided into smaller triangles until the monotonic criterion is satisfied. In fact, this is why breakpoints or points are often employed to assist triangulation in TIN. In contrast, bilinear interpolation in a rectangle does not guarantee that the terrain can be decomposed to monotonic patches, because a linear terrain structure (e.g., a ridge) does not always fall on the edge of a rectangle. This will be elaborated in a later subsection.

Monotonic terrain patch is a necessary condition for an interpolation method to be an isomorphism. In the following discussion, we shall assume that this necessary condition has been satisfied for TIN and linear interpolation in 1D. Based on this assumption, we examine whether topographic orderliness can be preserved by these methods. We start with TIN interpolation as its deviations are more straightforward and the conclusions obtained can assist the deviations of linear interpolation in 1D.

TIN Interpolation is an Isomorphism
Given two points A and B where \( z_A > z_B \) we examine whether \( Z_A > Z_B \) according to the following three scenarios:

1. A and B are on the same monotonic terrain patch which is modeled by triangle EFG (Figure 4a). Without loss of generality, supposing that E is the vertex with the highest elevation and FG is the edge facing E. From point A, a vertical line perpendicular to the horizontal plane can be drawn to obtain \( A' \) on triangle EFG and \( A_3 \) on the horizontal plane. \( A' \) and \( A_3 \) are called the projection of \( A \) on the triangle facet and the horizontal plane, respectively. It can be seen that \( A A_3 \) is the true elevation of \( A \), i.e., \( A A_3 = z_A \) and \( A_3 A' \) is the elevation interpolated by TIN, i.e., \( A_3 A_2 = Z_B \). Similar operation can be conducted on point B to obtain \( B' \) on triangle EFG and \( B_3 \) on the horizontal plane. Like point A, \( B B_3 = z_B \) and \( B_3 B' \). Since \( A' \) and \( B' \) are both in triangle EFG, they can be connected to form a line to meet edge FG or its extension at point C. C also has a projection on the horizontal plane which is denoted by \( C \). Figure 4 illustrates this process. The profile of the path from A to B is shown in Figure 4b. Note \( A'B'C'A_2B_3C_2 \) forms a trapezoid where:

\[
C_2' A_2 : C_2' B_2 = A_2' A_2 : B_2' B_2 = Z_A : Z_B.
\]  

\( (4) \)

**Figure 3.** Bilinear interpolation in a rectangle: (a) each terrain patch is approximated as a rectangle, and (b) interpolation inside a rectangle, \( s_A, s_B, s_C, \) and \( s_D \) are the areas of the sub-rectangles; \( s \) is the total area of the rectangle.

where \( \omega_{a}, \omega_{b}, \omega_{c}, \omega_{d} \) are the areal proportions of the sub-rectangles constructed using \( T \) (Figure 3). When \( T \) is on the edge of a rectangle, e.g., \( p \) in Figure 3a, bilinear interpolation is effectively reduced to linear interpolation on that edge. Because of this property, a point on the boundary of two rectangle patches always has a single interpolated elevation regardless of the patch on which interpolation is conducted. A one-to-one correspondence thus exists between the topographic surface and the bilinear surface. However, such a relationship cannot be guaranteed if bilinear interpolation is conducted on irregular quadrilaterals.

The above analysis shows that linear interpolation in 1D, TIN, and bilinear interpolation on a rectangle are all bijections. In the literature, higher-order patchwise polynomials such as the following 14-term bicubic function have also been proposed (Kidner, 2003):

\[
f(x, y) = a_{00} + a_{01}x + a_{02}y + a_{10}x^2 + a_{11}xy + a_{20}x^2y + a_{12}xy^2 + a_{21}x^2y^2 + a_{00}x^2y^2 + a_{01}x^2y + a_{02}x^2y^2 + a_{10}x^2y^2 + a_{11}x^2y + a_{12}x^2y^2 + a_{20}x^2y + a_{21}x^2y^2 + a_{22}x^2y^2.
\]

While higher-order polynomials may result in smaller interpolation error (Kidner, 2003; Shi and Tian, 2006), it has to be pointed out that these patchwise polynomials are not guaranteed bijective. In the case of the above bicubic interpolation, each patch is modeled as a bicubic surface and the 14 coefficients involved are estimated based on the reference points on that patch only. This creates the possibility that the bicubic functions for two contiguous patches may not be exactly the same. Both functions are equally valid to estimate the elevation of a boundary point, but the estimates may not always coincide. Since a location \( (x, y, z) \) may correspond to more than one \( (x, y, z) \), bijection is not guaranteed therefore the resultant DEM may not preserve topographic orderliness.

**Topographic Orderliness in Three Interpolation Methods**
Patchwise interpolation functions estimate the elevation of a point based on the reference data in the corresponding patch only. From this perspective, we call patchwise interpolation a local operation. On the other hand, topographic orderliness is a global property in the sense that any two points in the entire domain of a DEM, regardless whether they are located in the same patch, must satisfy that “\( z_A < z_B \) entails \( Z_A > Z_B \) and vice versa.” Whether and how a local operation results in the desired global property is the ultimate test to the effectiveness of an interpolation function.
In earlier discussion, we have assumed that each terrain patch is monotonic with no “bump” or “dip” exists inside.

Given that A and B are in the same monotonic patch and \( z_A > z_B \), the elevation from A to B must decrease monotonically. Since \( E \) is the vertex with the highest elevation, there should be \( C_1/A_1 > C_2/B_2 \). Per Equation 4 where \( C_1/A_1 > C_2/B_2 \), there must be \( Z_A > Z_B \). The proposition that \( z_A > z_B \) entails \( Z_A > Z_B \) is thus proved.

2. Point A and B are not in the same monotonic patch but in two patches adjacent to each other. Figure 5 are the two possible configurations under this scenario. In Figure 5a, the two triangle patches are not located within the same pair of contour lines. Since \( z_A > z_B \) and they are not on the same patch, they cannot fall between the same pair of contour lines. Instead, A must be between \( z_A \) and \( z_B \). B must be between \( z_B \) and \( z_A \). For point A, because the vertices of the triangle to which it belongs are located on \( z_A \) and \( z_B \), its interpolated elevation must be between \( z_A \) and \( z_B \). Similarly, the interpolated elevation of B must be between \( z_B \) and \( z_A \). i.e., \( z_A > Z_B > z_B > z_A \). Since \( z_A > z_B \) and \( z_B > z_A \), there must be \( Z_A > Z_B \). The proposition that \( z_A > z_B \) entails \( Z_A > Z_B \) is thus proved.

Figure 5b shows the other situation where the two triangle patches are bounded by the same pair of contour lines, i.e., \( z_0 > Z_0 > z_1 > z_1 \). Since the elevation from D \( (z_0 = z_1) \) to E \( (z_0 = z_1) \) changes continuously, there must exist a point \( C \) on path DE whose elevation satisfies \( z_0 > z_A > z_C > z_B > z_1 \). In fact, there exist innumerable such points. Given that A and C are on the same monotonic terrain patch, there should be \( z_A > Z_C \) according to the conclusion obtained in the previous scenario. Similarly, \( Z_C > Z_B \) Therefore, \( z_A > z_B \) entails \( Z_A > Z_B \).

3. Point A and B are located in two non-adjacent patches (Figure 6). Under such circumstance, there always exist a set of intermediate patches to connect the two non-adjacent patches. Derivation can be conducted in a manner similar to the same scenario-b. Assuming A and B are configured like that in Figure 6, i.e., they are between the same pair of contour lines. Since the elevation along boundary DE and DF changes continuously from \( z_0 \) to \( z_1 \), there must exist \( A_1, A_2, \ldots \) to satisfy \( z_{A_1} = z_{A_2} = z_{A_3} \). Similarly, a set of points \( B_1, B_2, \ldots \) can be located on the boundaries to satisfy \( z_{B_1} = z_{B_2} = z_{B_3} \). Let \( C_1 \) be a point between \( A_1 \) and \( B_1 \) on edge DE, there is \( z_{C_1} > z_{C_2} > z_{C_3} \). Since A and C are on the same patch, there should be \( Z_{A_1} > Z_{C_1} \). Repeat this process to find a point \( C_2 \) on boundary DF where \( z_{C_2} > z_{C_3} > z_{C_4} \). Since \( C_2 \) and \( C_3 \) are on the same monotonic patch, there must be \( Z_{C_2} > Z_{C_3} \). Similarly, \( Z_{C_1} > Z_{C_2} \). Combining these results together, there is \( Z_A > Z_B \).

In the case that A and B are not within the same pair of contour lines, the derivation is similar to that for Figure 5a. To avoid redundancy, we skip the detailed derivation. Combining the findings from all previous scenarios together, it can be concluded that TIN interpolation guarantees that \( z_A > z_B \) entails \( Z_A > Z_B \), therefore TIN is a homomorphism. As previously stated, we have also shown that TIN is bijective. These two aspects together suggests that TIN is a bijective homomorphism hence an isomorphism.
The isomorphism of TIN guarantees that TIN-interpolated DEMs preserve topographic orderliness. However, it has to be stressed that this conclusion hinges on the critical prerequisite that the terrain has been decomposed into monotonic patches through triangulation before TIN interpolation is applied. Unless the monotonic patch criterion is satisfied, there may not exist ancillary points such as \( C_1, C_2, \ldots \) in scenario-b on the boundary of the intermediate patches to facilitate the propagation of the inequalities. The one-to-one relationship between TIN mesh and the topographic surface is the most critical step in applying TIN to create monotonic patches. From this perspective, triangulation is the most critical step in applying TIN to create a DEM preserving topographic orderliness. The practical implications of this conclusion will be discussed in the Conclusions Section.

Linear Interpolation in 1D is an Isomorphism

Like TIN interpolation, topographic orderliness in linear interpolation can be examined in several scenarios. Let \( A \) and \( B \) be two DEM points where \( z_A > z_B \). If \( A \) and \( B \) are not located between the same pair of contour lines (Figure 7a), the inference is straightforward. Per Equation 1, there are \( z_A > Z_A > z_B \) and \( z_1 > Z_1 > z_2 \), therefore \( Z_A > Z_1 > Z_B \). The derivation becomes more complicated when \( A \) and \( B \) are within the same contour lines. To examine topographic orderliness, the topographic surface bounded by the contour lines needs to be divided into monotonic patches first. This is doable as the exact shape of a patch is not important as long as it is monotonic. Depending on how \( A \) and \( B \) are located with reference to the monotonic patches and the surface flow paths between the contour lines, the following scenarios exist:

1. \( A \) and \( B \) are on the same monotonic terrain patch, and they are on the same flow path (Figure 7b). Let Figure 7c be the profile of the flow path. In linear interpolation in 1D, this flow path is approximated by a straight line \( CD \) where \( z_C = z_0 \) and \( z_D = z_1 \). Let the projection of \( A \) and \( B \) on line \( CD \) be \( A' \) and \( B' \), respectively. Since surface water always flows down gradient, line \( CD \) must be monotonic. Given that \( A \) is higher than \( B \), there must be \( DA' > DB' \) or equivalently \( DA > DB \). In other words, \( DA'/DC > DB'/DC \). Per Equation 1, the interpolation elevation of \( A \) is:

\[
Z_A = \frac{DA}{DC} z_C + \left(1 - \frac{DA}{DC}\right) z_D
\]

Similarly, \( Z_B = \frac{DB}{DC} z_C + \left(1 - \frac{DB}{DC}\right) z_D \).

Since \( z_C > z_0 \) and \( DA'/DC > DB'/DC \), it can be shown with simple arithmetic processing that \( Z_A > Z_B \).

2. \( A \) and \( B \) are on the same monotonic terrain patch but not on the same flow path. A flow path passing through \( A \) can be constructed, which crosses the two contour lines at \( C_1 \) and \( D_1 \) respectively. Similarly, the flow path passing through \( B \) can be constructed and denoted as \( C_2D_2 \) in Figure 8a. Note \( z_{C_1} = z_B \) and \( z_{D_1} = z_A \). Figure 8b and Figure 8c show the profiles of these two flow paths. As previously explained, linear interpolation in 1D approximates a flow path as a straight line on which a terrain point has a projection. In Figure 8b, flow path \( C_1D_1 \) is approximated by the straight line \( C_1C_2 \) and \( A' \) is the projection of \( A \) on this straight line, therefore \( Z_A = z_{A'} \). Similarly, \( Z_B = z_{B'} \).

The elevation of \( A \) and \( B \) can also be estimated through TIN interpolation. In Figure 8a, \( C_1C_2D_2 \) are terrain points. They can be connected to form a flat triangle facet \( C_1C_2D_2 \). This triangle facet corresponds to a terrain patch which is bounded by segment \( C_1C_2 \) of contour line \( z_1 \), flow path \( C_2D_2 \), and path \( C_1D_2 \). Path \( C_1D_2 \) is constructed as follows: there are innumerable terrain points between \( C_1 \) and \( C_2 \) through each of these points.
Bilinear interpolation guarantees the one-to-one relationship in 1D if the point under interpolation is on the edge of a triangle patch. This suggests \( Z_B = Z_B^{TIN} \) where \( Z_B \) and \( Z_B^{TIN} \) are the estimated elevation of \( B \) by linear interpolation in 1D based on line \( CDB \) and TIN interpolation based on triangle \( CDB \), respectively. Similar inference can be applied to obtain \( Z_A = Z_A^{TIN} \).

The path \( CDB \) is monotonic therefore elevation descends continuously from \( Z_{C1} = z_0 \) to \( Z_{B0} = z_l \). Given \( z_0 > z_A > z_B > z_1 \), there must exist a point \( E \) on \( CDB \) to satisfy \( z_A > z_B > z_E \). A and \( E \) are both on the monotonic patch corresponding to triangle \( CDB \). If TIN is applied to estimate the elevation of \( A \) and \( E \), there must be \( Z_A > Z_E > Z_B \) because TIN is an isomorphism. Similarly, it can be shown that \( Z_B < Z_A < Z_E \). Combining the inequalities together, there is \( Z_B < Z_A < Z_E \). Since it has been shown that \( Z_B = Z_B^{TIN} \) and \( Z_A = Z_A^{TIN} \), there must be \( Z_B > Z_A > Z_E \).

3. \( A \) and \( B \) are not on the same monotonic patch. Under such a circumstance, the terrain between two consecutive contour lines can always be segmented into a set of monotonic patches. The two patches containing \( A \) and \( B \), respectively, can be connected through a series of intermediate monotonic patches. Let the contour line passing \( A \) crossing the boundary of these intermediate patches at \( A_i \), \( A_0 \), \ldots, \( A_l \). Similarly, let the contour line passing \( B \) crossing the boundary of the intermediate patches at \( B_i \), \( B_0 \), \ldots, \( B_l \). Let \( C_1 \) be a point between \( A_1 \) and \( B_1 \), i.e., \( z_{A_1} > z_{C_1} > z_{B_1} \), and \( C_2 \) a point between \( A_2 \) and \( B_2 \), i.e., \( z_{A_2} > z_{C_2} > z_{B_2} \). Furthermore, \( z_0 > Z_{C1} > Z_{C2} \). Since \( A \) and \( C_1 \) are on the same monotonic patch, there is \( Z_A > Z_{C1} \) according to the conclusion obtained previously. Similarly, there is \( Z_E > Z_{C2} \). Translating the inequalities, there is \( Z_A > Z_E \).

Combining this conclusion with those obtained previously, it can be concluded that topographical orderliness is guaranteed in linear interpolation in 1D.

**Bilinear Interpolation is not a Homomorphism**

Bilinear interpolation guarantees the one-to-one relationship between terrain and the bilinear surface only if each patch is a rectangle. However, unlike TIN and linear interpolation in 1D where the terrain can always be divided into monotonic patches, bilinear interpolation in a rectangle does not guarantee that the terrain surface can be divided into monotonic patches because breaklines do not always fall on the edge of a rectangle. Increasing the spatial resolution of the rectangle mesh improves the likelihood of a rectangle patch being monotonic, but can never guarantee it. This limitation determines that a DEM interpolated by bilinear interpolation can only preserve topographic orderliness partially.

**Discussion and Conclusions**

Topographic orderliness is an intrinsic characteristic of terrain. Any effort on building a model of the terrain (e.g., DEMs and topographic maps) must take it into account. Existing research on DEM accuracy has mainly focused on individual DEM points; the concept of topographic orderliness has never been discussed explicitly. In this paper, we conducted a theoretical examination on the relationship between topographic surface, DEM, and interpolation function using grid DEMs generated from contour lines as an example. It is pointed out that an interpolation function must be an isomorphism in order to result in a DEM preserving topographic orderliness. Isomorphism requires two necessary conditions: (a) the interpolation function must be monotonic, i.e., there exists a one-to-one relationship between the topographic surface and the surface created by the interpolation function, and (b) the interpolation function must be a homomorphism, i.e., \( z_0 < z_1 \) entails \( Z_0 < Z_1 \). The latter condition further entails that the interpolation function, if patchwise, must be able to divide the topographic surface into monotonic patches first. The ability to satisfy these conditions by three patchwise linear methods, namely linear interpolation in 1D, TIN, and bilinear interpolation, are examined. It is found that the one-to-one relationship can be satisfied by all three methods. However, only linear interpolation in 1D and TIN are homomorphisms because the quadrilateral patches used by bilinear interpolation are not guaranteed monotonic. This suggests that a DEM generated by linear interpolation in 1D or TIN can preserve topographic orderliness, but that by bilinear interpolation does not.

While linear interpolation in 1D and TIN are both isomorphisms in theory, the challenges to satisfy the two aforementioned necessary conditions in each method are rather different. In TIN interpolation, the difficulty lies in triangulation which is the process to construct the triangle facets. Recall that each triangle facet must be monotonic in order to achieve isomorphism. Unfortunately, this is not automatically guaranteed by methods such as Delaunay triangulation. Breaklines and break points which denote important terrain features must be incorporated. Since isomorphism requires that every triangle patch must be

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**Figure 8. Isomorphism in linear interpolation in 1D:** (a) A and B are on the same monotonic terrain patch but not the same flow path, (b) profile of the flow path passing through A, and (c) profile of the flow path passing through B.
monotonic, it means that all breaklines and points which are deemed necessary at a given geographic scale must be extracted rigorously. This requirement has some interesting implications. From the perspective of cost effectiveness, the extraction of breaklines and points is somewhat redundant when a DEM from a topographic map because break lines and points are already embedded in the contour lines and spot elevations. An intelligent interpolation method should be able to recognize the breaklines and points automatically and respect them proactively, instead of relying on external enforcement. From the implementation perspective, the extraction of break lines and points is also not easy. While much research on this topic has been conducted since the 1970s, an optimal algorithm is still under exploration. Until this challenge is resolved, the practical utility of TIN to generate a DEM preserving topographic orderliness will remain limited.

Linear interpolation in 1D does not require breaklines or points because its interpolation is conducted on surface flow paths which are always monotonic. In fact, Equation 1 guarantees that the interpolation will never surpass important terrain features such as ridges, hilltops, or valleys. The challenge in linear interpolation in 1D lies in the delineation of the flow paths between two contour lines. Although many algorithms have been proposed in the literature, some implementations are only close approximations (Hu et al., 2007). One rigorous solution is to generate the intermediate contour lines (e.g., half-interval, quarter-interval, etc.) recursively through the computation of the Voronoi boundary between the original contour lines (Hu et al., 2007). Compared to TIN, linear interpolation in 1D preserves topographic orderliness proactively. Additionally, research based on numeric simulation has found that linear interpolation in 1D results in smaller interpolation error than TIN and bilinear interpolation (Hu et al., 2009). Based on these reasons, we recommend linear interpolation in 1D as the preferred method to interpolate a DEM.

It has to be pointed out that the conclusions on the isomorphism of the three interpolation methods are obtained under the assumption that the source data is error free. In reality, source data usually have at least some random error. These errors can be propagated to a DEM point through the interpolation function. According to Hu et al. (2009), the actual elevation of a DEM point can be written as:

\[ z_T = z_s + (\delta_T + R_T) \]

where \( z_T \) is the true unknown elevation at \( T \), \( R_T \) is the error due to the interpolation function utilized, and \( \delta_T \) is the impact of the error in the source data on \( T \). This paper has assumed that \( \delta_T = 0 \) to prove that \( z_T > z_s \) entails \( z_s + R_s > z_s + R_T \) in TIN and linear interpolation in 1D. In reality, \( \delta_T \) may not be zero, and its impact varies from point to point. When \( \delta_T \) is factored into the equation, \( Z_s > Z_s \) does not necessarily hold. This highlights the importance of high quality source data in order to generate a DEM preserving topographic orderliness.

**Directions for Future Research**

While topographic orderliness offers a new angle to DEM accuracy assessment, it has to be pointed out that the research in this paper still focuses on the DEM point scale as does most existing research. The overall structure of the terrain, which is usually more important to practical applications, has not been taken into account. Such an omission may result in serious information loss to the DEM users. By definition, the purpose of a DEM is to provide an effective description of a topographic surface. Features such as hilltops, saddles, depressions, and even the drainage patterns are fundamentally important to inform the DEM users about the terrain. Due to the grid structure of most DEMs, these important terrain features do not always fall on a DEM point (or a set of DEM points). Consequently, they can be lost in the crude DEM which is the direct output from the interpolation. Such a crude DEM, even it has minimal interpolation error and preserves topographic orderliness, cannot be considered of high accuracy because of its inability to reflect the overall structure of the terrain.

Thus entails the necessity of DEM generalization which is after cartographic generalization and refers to the post-processing of a coarse DEM so as to retain the important terrain features. In the literature, only limited research has been conducted on this subject. An appropriate methodology is yet to be established. We envision that DEM generalization includes three steps. The first is to generate a crude DEM which preserves topographic orderliness. This serves as the basis of DEM generalization. The second is to identify the terrain features which must be retained at the given geographic scale or DEM resolution. The importance of a feature depends on the resolution of a DEM. A small pit may need to be retained in a fine-resolution DEM but can be omitted in a coarse-resolution DEM. Also in this step, a method to rank the importance or priority of different features needs to be developed. In the case that two features fall in the same grid cell, this ranking method can then be applied to decide which feature to retain. The last step is to decide whether and how to interpolate. Such a crude DEM, even it has minimal interpolation error and preserves topographic orderliness, cannot be considered of high accuracy because of its inability to reflect the overall structure of the terrain.

**References**


