LAB #1: TIME AND FREQUENCY RESPONSES OF SERIES RLC CIRCUITS

Objective:
To investigate the step, impulse, and frequency responses of series RLC circuits. To compare experimental results with theory and PSpice simulations, and to account for possible differences. To gain familiarity with Bode plots.

Components:
1 × 3.9-mH inductor, 1 × 10-nF capacitor, 1 × 1 kΩ potentiometer, and resistors: 1 × 10 Ω, 1 × 39 Ω, 1 × 1.0 kΩ, and 1 × 2.0 kΩ, 1 × 3.9 kΩ, (all 5%, ¼ W).

Instrumentation:
An RLC meter, a waveform generator (square-wave, pulse, and sine-wave), and a dual-trace oscilloscope.

References:

PART I – THEORETICAL BACKGROUND

RLC circuits are classical examples of second-order systems. Together with their mass-spring-dashpot mechanical analog, they are used to illustrate fundamental systems-theory concepts and techniques, such as Laplace-transform techniques and resonance.

The current response of the series RLC circuit of Fig. 1 is found via Laplace-transform techniques as $I(s) = Y(s)V(s)$, where $I(s)$ and $V(s)$ are the Laplace transforms of $i(t)$ and $v(t)$, $s$ is the complex frequency, and $Y(s)$ is the complex admittance, that is, the reciprocal of the complex impedance $Z(s)$,

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{R + sL + \frac{1}{sC}} = \frac{sC}{s^2LC + sRC + 1}$$

This function is usually expressed in the standardized form

$$Y(s) = \frac{1}{R \times \frac{2\zeta (s/\omega_0)}{(s/\omega_0)^2 + 2\zeta (s/\omega_0) + 1}}$$

Fig. 1 – Series RLC circuit.
\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \zeta = \frac{R}{2} \sqrt{\frac{C}{L}} \]  

(2)

Here, \( \omega_0 \) is the undamped natural frequency, in rad/s, and \( \zeta \) is the damping ratio, dimensionless. The values of these parameters are set by those of the components making up the circuit.

The values of \( s \) for which the denominator of \( Y(s) \) becomes zero are called the poles of \( Y(s) \), and therefore, the zeros of \( Z(s) \). They are easily found to be

\[ s_{1,2} = \omega_0 \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \]  

(3)

We have the following significant cases:

- For \( \zeta > 1 \), the poles are real and distinct, and the system is said to be overdamped.
- For \( \zeta = 1 \), the poles are real and coincident, and the system is said to be critically damped.
- For \( 0 < \zeta < 1 \), the poles are complex conjugate, or

\[ s_{1,2} = \omega_0 \left( -\zeta \pm j\sqrt{1-\zeta^2} \right), \text{ where } j^2 = -1. \]  

The system is now said to be underdamped.

In each of the above cases the poles lie in the left-half of the complex plane \( s \). For \( \zeta = 0 \), the poles lie right on the imaginary axis, and the system is said to be undamped. It is apparent that varying \( R \) while keeping \( L \) and \( C \) constant will move the poles around in the complex plane.

Systems theory indicates that the response \( i(t) \) to a given excitation \( v(t) \) can be found as \( i(t) = L^{-1}\{I(s)\} = L^{-1}\{Y(s)V(s)\} \), where \( L^{-1} \) indicates inverse Laplace transformation. The responses of greatest practical interest in engineering are the impulse, the step, and the ac or frequency responses. The current response \( i(t) \) is readily visualized with the oscilloscope by observing the voltage \( v_R(t) \) across the resistance \( R \); then, \( i(t) = v_R(t)/R \). Of great interest are also the capacitance and inductance responses \( v_C(t) \) and \( v_L(t) \).

All responses can readily be visualized via PSpice. The PSpice circuit of Fig. 2 displays the step response across \( C \) for the case \( \omega_0 = 1 \) rad/s. You can simulate this circuit on your own by downloading the appropriate files from the Web. To this end, go to

http://online.sfsu.edu/~sfranco/CoursesAndLabs/Labs/301Labs.html

Once there, click on PSpice Examples and follow the instructions contained in the Readme file.

The Transient Response:

Figure 3 shows the step or transient response across \( C \) for three different values of \( \zeta \). It can be proved that for \( 0 < \zeta < 1 \), this response is a damped sinusoid with the frequency
\[ \omega_d = \omega_0 \sqrt{1 - \zeta^2} \]  

called the damped frequency. We also observe the presence of overshoot, defined as

\[ \text{OS(\%)} = 100 \frac{v_{O(\text{peak})} - v_{O(\infty)}}{v_{O(\infty)}} \]

where \( v_{O(\infty)} \) is the value of \( v_O \) in the limit \( t \to \infty \). The overshoot is related to \( \zeta \) as

\[ \text{OS(\%)} = 100 e^{-\pi \zeta \sqrt{1 - \zeta^2}} \]  

The smaller the value of \( \zeta \), the higher the overshoot and the longer it takes for the oscillation to die out. In the limit \( \zeta \to 0 \) we have a sustained oscillation with undamped natural frequency \( \omega_0 \). If \( \zeta \) is gradually increased from zero, the oscillation will die out more and more rapidly until the point is reached where there will be no more oscillation. This point corresponds to critical damping, or \( \zeta = 1 \). For \( \zeta > 1 \), not only is there no oscillation, but the system takes even a longer time to reach its steady state.

**Frequency Response:**

Systems theory indicates that the frequency response of a circuit is found by letting \( \zeta \to j \omega \) in its transfer function. In this case it is also more common to work with the parameter \( Q = 1/(2\zeta) \), after which our expression above becomes

\[ Y(j\omega) = \frac{1}{R} \times H_{BP}(j\omega) \]

where

\[ H_{BP}(j\omega) = \frac{j(\omega/\omega_0)}{1 - (\omega/\omega_0)^2 + j(\omega/\omega_0)/Q} \]  

Fig. 3 – Step or transient response across \( C \) for different values of \( \zeta \).
\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad Q = \frac{1}{R \sqrt{LC}} \] (7)

The function \( H_{BP}(\omega) \) is called the **standard second-order band-pass function**.

To investigate the frequency response of our circuit, we apply an ac voltage of the type

\[ v_i(t) = V_{im} \cos \omega t \] (8a)

and we observe the response \( v_o(t) = R i(t) \) across the resistor, which is an ac voltage of the type

\[ v_o(t) = V_{om} \cos(\omega t + \phi) \] (8b)

Here, \( V_{im} \) and \( V_{om} \) are the **peak amplitudes** (in V), \( \omega \) is the **angular frequency** (in rad/s), and \( \phi \) is the **phase angle** (in degrees). The parameters of the response are related to those of the applied voltage as

\[ V_{om} = |H_{BP}| \times V_{im} \quad \phi = \angle H_{BP} \] (9)

where \( |H_{BP}| \) and \( \angle H_{BP} \) are, respectively, the **magnitude** and **phase** of \( H_{BP} \).

The PSpice circuit of Fig. 4 is used to visualize the **frequency response** across \( R \) for the case \( \omega_0 = 1 \) rad/s. Again, you can simulate this circuit on your own by downloading its files from the Web, as mentioned earlier. Figure 5 shows the logarithmic plots of **magnitude** and **phase**, also called **Bode plots**, for three different values of \( Q \). Each magnitude curve peaks at 0 dB for \( \omega = \omega_0 \), this being the reason why \( \omega_0 \) is also called the **resonance frequency**. Moreover, each curve drops to -3 dB at two frequencies \( \omega_L \) and \( \omega_H \) such that

\[ \omega_L = \omega_0 \left( \frac{1}{\sqrt{1 + \frac{1}{4Q^2} - \frac{1}{2Q}}} \right) \quad \omega_H = \omega_0 \left( \frac{1}{\sqrt{1 + \frac{1}{4Q^2} + \frac{1}{2Q}}} \right) \] (10)

It is readily seen that these frequencies satisfy the condition \( \omega_L \times \omega_H = \omega_0^2 \), and that phase is \( \pm 45^\circ \) at these frequencies. Moreover, the **half-power bandwidth**, defined as \( BW = \omega_H - \omega_L \), is such that

\[ Q = \frac{\omega_0}{BW} \] (11)

It is apparent that the narrower the \( BW \) for a given \( \omega_0 \), the higher the value of \( Q \). Consequently, \( Q \) provides a measure of the degree of **selectivity** of a filter such as ours.

![Fig. 4 – PSpice circuit to display the frequency response across R.](image-url)
PART II – EXPERIMENTAL PART

To lower the output resistance $R_s$ of the function generator (usually an ill-defined parameter on the order of 50 $\Omega$) to a smaller and more predictable value, we interpose a voltage-divider adaptor as shown in Fig. 6. Note that because of the internal ground connection of the oscilloscope, we must arrange the elements so that the element across which we wish to observe the response is always located at the site denoted as $X_3$.

By Thevenin’s theorem, the circuit reduces to the equivalent of Fig. 7 for the case in which we observe the response across the capacitor. Here, $v_{OC}$ and $R_{eq}$ are the parameters of the equivalent source, $R_L$ is the winding resistance of the coil, and $R_p$ is a variable resistance that we adjust to achieve specific values of $\zeta$ (or $Q$) for our circuit. This variable resistance is implemented via a potentiometer with the wiper connected to either one of its remaining terminals. If you need a value of $R_p$ greater than the potentiometer’s rating, use a suitable resistance in series. The expressions for $\zeta$ and $Q$ derived above still

![Fig. 5 – Bode plots for different values of Q](image-url)

![Fig. 6 – Experimental setup.](image-url)
hold, provided we use

\[ R = R_{eq} + R_L + R_p \]  \hspace{1cm} (11)

In our case, \( R_{eq} \cong (50 + 39)/10 \cong 9 \Omega \), \( R_L \) is measured, and is \( R_p \) is adjusted to specific values found via calculation, as we shall see shortly.

**Initial Measurements and Calculations:**

Henceforth, steps shall be identified by letters as follows: C for calculations, M for measurements, S for SPICE simulation.

**M1:** Using an \( RLC \) meter from the stockroom, measure and record the values of \( C, L, \) and \( R_L \) (the resistance of the coil). By how much do \( L \) and \( C \) differ from their nominal values?

**Remark:** Each measurement must be expressed in the form \( X \pm \Delta X \) (e.g. \( C = 8.7 \pm 0.05 \text{ nF} \)), where \( \Delta X \) represents the estimated uncertainty of your measurement, something you have to figure out based on measurement concepts and techniques learned in Engr 206 and Engr 300.

**C2:** Calculate the undamped natural frequency \( f_0 = 1/(2\pi \sqrt{LC}) \).

**Remark:** Again, you must express your result in the form \( f_0 \pm \Delta f_0 \), where the uncertainty \( \Delta f_0 \) stems from the uncertainties \( \Delta C \) and \( \Delta L \) of Step M1. Just writing something like \( f_0 = 934.2238 \text{ Hz} \) will be considered meaningless.

**Step Response:**

**C3:** Calculate the three values of \( R \), and hence of \( R_p = R - R_{eq} - R_L \), that result in \( \zeta = 5, \zeta = 1, \) and \( \zeta = 0.2 \), with the values of \( L \) and \( C \) measured in Step M1. For the case \( \zeta = 0.2 \), compute also the damped frequency \( f_d \) and overshoot \( OS(\%) \) via Eqs. (4) and (5).

**C4:** Calculate the values of the poles for the three specified values of \( \zeta \), and show their complex-plane locations. Be neat and precise.

**S5:** Using PSpice, along with the component values of Steps M1 and C3, plot the response of the circuit of Fig. 7 to a 1-V step for the three specified values of \( \zeta \). For the case \( \zeta = 0.2 \), use the cursor facility of PSpice to estimate the overshoot \( OS(\%) \) as well as the period \( T_d \) of the decaying oscillation and, hence, the damped frequency \( f_d = 1/T_d \). Compare with the predicted values in Step C3, comment.

**M6:** Assemble the circuit of Fig. 6, with the coil as \( X_1 \), the potentiometer as \( X_2 \), and the capacitor as \( X_3 \),

\[ \begin{array}{ccc}
\text{Coil} & \text{Pot} & \text{Cap} \\
R_{eq} & R_L & R_p \\
L & & C \\
\text{v}_1 & \text{v}_2 & \text{v}_0 \end{array} \]

**Fig. 7** – Equivalent circuit of Fig. 6 for the case in which \( X_3 \) is the capacitor (\( R_{eq} = 9 \Omega \))
so that its equivalent is as in Fig. 7. Keeping in mind that \( R_{eq} = 9 \Omega \), adjust \( R_p \) for \( \zeta = 1 \). Then, while monitoring \( v_1 \) with Ch.1 of the oscilloscope set on DC, adjust the waveform generator so that \( v_1 \) is a square wave alternating between 0 V and 1 V with a period of about 10/\( f_0 \), where \( f_0 \) is the undamped frequency of Step C2 (make sure you know where the 0-V baseline is on the screen!)

Now, observe and record the circuit’s response by monitoring \( v_2 \) with Ch. 2 of the oscilloscope set on DC. Finally, compare with the response predicted via PSpice in Step S5, and account for any differences.

**Note:** In this and the subsequent steps, if the pot is not sufficient to achieve the desired resistance value, use suitable combinations of resistances in series with the pot. For instance, connecting the 1-kΩ pot in series with a 3 kΩ ordinary resistor will allow you to span the range of 3 kΩ to 4 kΩ.

**M7:** Repeat step M6, but with \( R_p \) adjusted for \( \zeta = 5 \). Provide a physical justification for why the response is now so sluggish.

**M8:** Repeat step M6, but with \( R_p \) adjusted for \( \zeta = 0.2 \). Also, from the oscilloscope trace, estimate \( f_d \) and OS(%) in a manner similar to Step S5, compare with those predicted in Step C3, and comment. Finally, provide a physical justification for why the response is now oscillatory.

**Impulse Response:**
Leaving the potentiometer setting as in Step M8 (\( \zeta = 0.2 \)), interchange \( R_p \) and \( C \) so that the circuit becomes as in Fig. 8. Then, change the waveform generator settings so that \( v_1 \) is now a pulse train consisting of pulses each alternating between 0 V and 1 V with a pulse-width of about 0.1/\( f_0 \), where \( f_0 \) is the undamped frequency calculated in Step C2. A pulse this narrow will provide a good approximation to the impulse function for our circuit. Moreover, to be able to see a repetitive trace on the oscilloscope, adjust the waveform generator settings so that the above pulses repeat with a frequency of about 10/\( f_0 \).

**M9:** While triggering the oscilloscope from \( v_1 \), observe \( v_2 \) with the other channel and record it (for best visualization, you may need to adjust the repetition frequency from the initial suggested value of 10/\( f_0 \).) Next, measure the period \( T_d \) of the damped oscillation, calculate \( 1/T_d \), and compare with \( f_d \) of Step C3. Finally, justify the waveform for the response \( v_2 \) using physical insight.

**Frequency Response:**
To investigate this type of response we still use the circuit of Fig. 8, except that we change the waveform generator settings so that \( v_1 \) is now a sinusoidal signal with a constant peak amplitude of 1 V, 0-V DC, and variable frequency \( f \).

You can measure \( f \) by \( (a) \) reading the frequency setting on the waveform generator, or \( (b) \) by measuring the period \( T \) with the oscilloscope and then computing \( f = 1/T \), or \( (c) \) by using a frequency meter from the stockroom. It is up to each group to decide which method to pursue, and to justify your choice in the final report. No matter which method you use, your data must always be expressed in the form \( f \pm \Delta f \), as mentioned above.

**C10:** Find the value of \( R \), and, hence, of \( R_p \), that results in \( Q = 5 \) in the circuit of Fig. 8. Then, using Eqs. (10) and (11), calculate \( f_L, f_H \), and the bandwidth \( BW \). As usual, express your results in the form \( X \pm \Delta X \).

**S11:** Using PSpice, along with the component values of Steps M1 and C10, generate the Bode Plots of the circuit of Fig. 8. Then, using the cursor facility of PSpice, estimate \( f_i \) and \( f_{th} \) first as the -3-dB frequencies on the magnitude plot, then as the ±45° frequencies on the phase plot. Compare the resulting values of \( f_i \), \( f_{th} \), and \( BW \) against those of Step C10, and account for possible differences.

**M12:** While monitoring \( v_2 \) with the oscilloscope, vary the waveform generator’s frequency \( f \) until \( v_2 \)
reaches its maximum, and record the value of $f$. This is the experimental value of $f_0$ (express it in the form $f_0 \pm \Delta f_0$). Compare with the calculated value of Step C2. Do they agree within their respective uncertainties? Account for possible differences! Also, how does the maximum amplitude of $v_2$ compare with the amplitude of $v_1$? Justify via suitable voltage-divider calculations!

**M13:** Vary the waveform generator’s frequency $f$ until the amplitude of $v_2$ is down to 70.7% of its maximum as found in Step M12. There are two such frequencies, namely, $f_L$ and $f_H$. How do they compare with the calculated values of Step C10. Do they agree within their respective uncertainties?

**M14:** Repeat Step M13, except that now we shall find $f_L$ and $f_H$ as the $\pm 45^\circ$ frequencies. For phase measurements, use Channel 1 and Channel 2 of the oscilloscope for input and output. Which of the methods of estimating $f_L$ and $f_H$ do you think is the most and which the least dependable, and why?

**M15:** Verify experimentally the following important properties:

- For $f \ll f_0$, increasing $f$ by a factor of 10 increases amplitude also by a factor of 10, this being the reason why it is said that the slope of the magnitude curve is $\pm 20$ dB/dec there.
- For $f \gg f_0$, increasing $f$ by a factor of 10 decreases amplitude also by a factor of 10, this being the reason why the slope of the magnitude curve is said to be $-20$ dB/dec there.

**M16:** Interchange $R_p$ and $C$ so that we are back to the circuit of Fig. 7, to observe the response across $C$. Find the value of $R$ that results in $Q = 1/\sqrt{2} = 0.707$, and hence adjust $R_p$ accordingly. Then, by suitably varying the waveform generator’s frequency $f$ while leaving amplitude and DC offset unchanged, find experimentally the following:

- The $-3$-dB frequency $f_{3dB}$
- The low-frequency amplitude of $v_2$
- The amplitude of $v_2$ at $10f_{3dB}$ and $100f_{3dB}$

**C17:** Using the data of Step M16, construct the experimental magnitude Bode plot of the response across the capacitor. Hence, justify the designation second-order low-pass response. What is the slope for $f \gg f_0$, in dB/dec?

**S18:** Using PSpice, plot the magnitude response across $C$. Then, compare with the plot of Step C17, and account for possible difference.

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**Fig. 8 – Circuit to investigate the impulse and frequency responses across $R_p$.**