Chapter 2
Metric Spaces

The purpose of this chapter is to present a summary of some basic properties of metric and topological spaces that play an important role in the main body of the book.

2.1 Metrics and Pseudometrics

Definition 2.1. A metric space is a pair \((X, d)\) where \(X\) is a nonempty set and \(d\) is a function \(d : X \times X \rightarrow \mathbb{R}\) satisfying conditions:

- (M1) \(d(x, y) = 0\) if and only if \(x = y\),
- (M2) \(d(x, y) = d(y, x)\), symmetry
- (M3) \(d(x, y) + d(y, z) \geq d(x, z)\), triangle inequality

for all \(x, y, z \in X\). Elements of the set \(X\) are called points. The number \(d(x, y)\) is said to be the distance between points \(x\) and \(y\). The function \(d\) is called a metric or a distance function.

Note that conditions M1–M3 imply that the distance function \(d\) is necessarily nonnegative. Indeed,

\[
0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y),
\]

for all \(x, y \in X\).

For any three points \(x, y, z\) in a metric space \((X, d)\), the triangle inequality (M3) claims that each of the three distances \(d(x, y), d(y, z)\), and \(d(z, x)\) is not greater than the sum of two other distances:

\[
\begin{align*}
d(x, y) &\leq d(x, z) + d(z, y), \\
d(y, z) &\leq d(y, x) + d(x, z), \\
d(z, x) &\leq d(z, y) + d(y, x).
\end{align*}
\]
This system of three inequalities is equivalent to the following chain of inequalities:

\[ |d(x, z) - d(y, z)| \leq d(x, y) \leq d(x, z) + d(z, y) \]  \hspace{1cm} (2.1)

(cf. Exercise 2.2).

By induction, the triangle inequality is generalized to \( n > 3 \) points as

\[ d(x_1, x_n) \leq d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \]  \hspace{1cm} (2.2)

(cf. Exercise 2.4).

We will often refer to inequalities (2.1) and (2.2) as “triangle inequalities”. The name is motivated by theorems in the Euclidean geometry (cf. drawing in Fig. 2.1).

![Fig. 2.1 Triangle on the Euclidean plane.](image)

**Example 2.1.** The set \( \mathbb{R} \) of all real numbers endowed with the distance function

\[ d(x, y) = |x - y|, \]

where \(|x|\) is the absolute value of \( x \), is a metric space. Similarly, the set of all complex numbers \( \mathbb{C} \) is a metric space with the distance function

\[ d(z, w) = |z - w|, \]

where \(|z|\) is the modulus (absolute value) of \( z \) in \( \mathbb{C} \) (cf. (1.5)).

**Example 2.2.** Let \( X \) be a nonempty set. It is easily seen that the function

\[ d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y, \end{cases} \]

is a metric (also known as a trivial metric). The space \((X, d)\) is called the discrete metric space on \( X \).
Example 2.3. Let $X$ be a vector space over field $F$ (which is either $\mathbb{R}$ or $\mathbb{C}$). A norm on $X$ is a real-valued function on $X$, whose value at $x \in X$ is denoted by $\|x\|$, with the properties:

a) $\|x\| = 0$ if and only if $x = 0$,

b) $\|\alpha x\| = |\alpha|\|x\|$, 

c) $\|x + y\| \leq \|x\| + \|y\|$, 

for all $x, y \in X$ and $\alpha \in F$. Note that these conditions imply that norm is a nonnegative function on $X$. It can be readily verified that $d(x, y) = \|x - y\|$ is a metric on $X$.

Let $(X, d)$ be a metric space. It is clear that for a nonempty subset $Y$ of $X$ the restriction of $d$ to the set $Y \times Y$ is a metric on $Y$. Usually this metric is denoted by the same symbol $d$. The metric space $(Y, d)$ is called a subspace of the space $(X, d)$.

Example 2.4. Any nonempty set of real numbers is a metric space with the distance function given by $|x - y|$ (cf. Example 2.1). In particular, any interval in $\mathbb{R}$ is a metric space as well as the set $\mathbb{Q}$ of all rational numbers.

It is a usual custom to omit reference to the metric $d$ in the notation $(X, d)$ and write “a metric space $X$” instead of “a metric space $(X, d)$”. However, we use the latter notation if different metrics on the same set are considered.

A more general concept of the distance than the one of a metric is especially useful in functional analysis.

Definition 2.2. A pseudometric on a nonempty set $X$ is a real-valued function $d$ satisfying conditions:

$$(\text{PM1}) \quad d(x, x) = 0,$$

$$(\text{PM2}) \quad d(x, y) = d(y, x), \quad \text{symmetry}$$

$$(\text{PM3}) \quad d(x, y) + d(y, z) \geq d(x, z), \quad \text{triangle inequality}$$

for all $x, y, z \in X$ (cf. Definition 2.1). The pair $(X, d)$ is said to be a pseudometric space.

Evidently, a metric space is also a pseudometric space such that the distance between two distinct points is a positive number.

Example 2.5. Consider the following two functions on the set $\mathbb{C}$:

$$d_1(z, w) = |\text{Re}(z) - \text{Re}(w)| \quad \text{and} \quad d_2(z, w) = |\text{Im}(z) - \text{Im}(w)|.$$ 

Both functions are pseudometrics on $\mathbb{C}$. Note, for instance, that $d_2(x, y) = 0$ for any real numbers $x$ and $y$.

Example 2.6. Let $X$ be a vector space over the field $F$. A function $p : X \to \mathbb{R}$ is called a seminorm on $X$ if it satisfies conditions:
a) \( p(\alpha x) = |\alpha|p(x) \),
b) \( p(x + y) \leq p(x) + p(y) \),

for all \( \alpha \in \mathbb{F} \) and \( x, y \in X \). It can be shown (cf. Exercise 2.7) that the function

\[
d(x, y) = p(x - y)
\]

is a pseudometric on \( X \).

### 2.2 Open and Closed Sets

**Definition 2.3.** Let \( X \) be a metric space, \( x \in X \), and \( r > 0 \). The set

\[
B(x, r) = \{ y \in X : d(y, x) < r \}
\]

is called an **open ball of radius** \( r \) **centered at** \( x \). Similarly, a **closed ball of radius** \( r \) **centered at** \( x \) is the set

\[
\overline{B}(x, r) = \{ y \in X : d(y, x) \leq r \}.
\]

The definitions of open and closed balls have their roots in the Euclidean geometry. However, in abstract metric spaces, these concepts may have some counterintuitive properties (cf. Exercises 2.11 and 2.13).

An open ball of radius \( \varepsilon > 0 \) centered at \( x \in X \) is called an **\( \varepsilon \)-neighborhood** of \( x \). A **neighborhood** of \( x \in X \) is any subset of the space \( X \) containing the \( \varepsilon \)-neighborhood of \( x \) for some \( \varepsilon > 0 \).

Let \( E \) be a subset of a metric space \( X \). A point \( x \in E \) is said to be an **interior point of** \( E \) if \( E \) contains an open ball centered at \( x \). The **interior of the set** \( E \) is the set of all its interior points. This set is denoted by \( \text{int} \, E \).

**Example 2.7.** In these examples, all sets under consideration are subsets of the metric space \( \mathbb{R} \).

a) The interior of an open interval \( (a, b) \) is the interval itself.
b) The interior of the closed interval \( [0, 1] \) is the open interval \( (0, 1) \).
c) The interior of the set of rational numbers \( \mathbb{Q} \) is empty (cf. Exercise 2.16).

**Definition 2.4.** A subset \( U \) of a metric space \( X \) is said to be **open** if it contains an open ball centered at each of its points.

In other words, a subset \( U \) of \( X \) is an open set if it coincides with its interior.

Any metric space \( X \) has at least two distinct open subsets, namely, the empty set and the set \( X \) itself. If the metric space \( X \) consists of a single point, then \( \emptyset \) and \( X \) are the only open subsets of \( X \) (cf. Exercise 2.17).

Two fundamental properties of open sets in a metric space are found in the next theorem.
2.2 Open and Closed Sets

**Theorem 2.1.** Let $X$ be a metric space and $\mathcal{T}$ the collection of all open sub-sets of $X$. Then

a) The union of any family $\{U_i\}_{i \in J}$ of open sets is an open set:

$$\bigcup_{i \in J} U_i \in \mathcal{T}.$$ 

b) The intersection of any finite family $\{U_1, \ldots, U_n\}$ of open sets is an open set:

$$\bigcap_{i=1}^n U_i \in \mathcal{T}.$$ 

**Proof.**

a) If a point $x$ belongs to the union $\bigcup_{i \in J} U_i$, then there is $i \in J$ such that $x \in U_i$. Because $U_i$ is an open set, there is an open ball $B(x, r)$ that is a subset of $U_i$. Clearly, $B(x, r) \subseteq \bigcup_{i \in J} U_i$. Hence, $\bigcup_{i \in J} U_i$ is an open set.

b) It suffices to show that the intersection $U_1 \cap U_2$ of two open sets is open. If this intersection is empty, then we are done because the empty set is open. Otherwise, let $x \in U_1 \cap U_2$. Inasmuch as the sets $U_1$ and $U_2$ are open, there are balls $B(x, r_1) \subseteq U_1$ and $B(x, r_2) \subseteq U_2$. Then the ball $B(x, r)$ with $r = \min\{r_1, r_2\}$ belongs to the intersection $U_1 \cap U_2$, which is the desired result. □

Let $E$ be a subset of a metric space $X$. A point $x \in X$ is said to be a **limit point of** $E$ (or an **accumulation point of** $E$) if every neighborhood of $x$ contains a point in $E$ distinct from $x$. The set consisting of the points of $E$ and the limit points of $E$ is called the **closure** of $E$ and denoted by $\overline{E}$.

**Definition 2.5.** A subset $F$ of a metric space $X$ is said to be **closed** if it contains all its limit points.

Hence, a subset $F$ of $X$ is closed if it coincides with its closure, $F = \overline{F}$.

**Example 2.8.** a) The open unit ball in the complex plane,

$$B(0, 1) = \{z \in \mathbb{C} : |z| < 1\},$$

is an open set.

b) The closed unit ball in the complex plane,

$$\overline{B}(0, 1) = \{z \in \mathbb{C} : |z| \leq 1\},$$

is a closed set which is the closure of $B(0, 1)$,

$$\overline{B}(0, 1) = \overline{B}(0, 1).$$

c) Let $X$ be a discrete metric space of cardinality greater than 1 and $x \in X$. Then
so the closure of an open ball $B(x, r)$ in a metric space can be a proper subset of the closed ball $\overline{B}(x, r)$ (cf Exercise 2.18).

The following theorem establishes a close relationship of the concepts of open and closed sets.

**Theorem 2.2.** A subset $E$ of a metric space $X$ is closed if and only if its complement $E^c$ is open.

**Proof.** It is easy to see that the result holds if $E = \emptyset$ or $E = X$. Thus we assume that $E$ is a nonempty subset of $X$ which is different from $X$.

(Necessity.) Let $E$ be a closed set and $x$ an element of $E^c = X \setminus E$. Because $x \notin E$, it is not a limit point of $E$. Hence, there is a neighborhood of $x$ which is a subset of $E^c$. It follows that $E^c$ is an open set.

(Sufficiency.) Suppose that the complement $E^c$ of the set $E$ is open and let $x$ be a limit point of $E$. Because every neighborhood of $x$ contains a point of $E$ different from $x$, the point $x$ does not belong to the open set $E^c$. Therefore, $x \in E$, that is, $E$ is a closed set. □

By applying Theorem 2.2 to properties a) and b) of open sets established in Theorem 2.1, we obtain fundamental properties of closed sets.

**Theorem 2.3.** In a metric space,

a) the union of a finite family of closed sets is a closed set,

b) the intersection of any family of closed sets is a closed set.

The proof of the next theorem is left as an exercise (cf. Exercise 2.19).

**Theorem 2.4.** The interior of a subset $E$ of a metric space is the maximum open set contained in $E$. The closure of $E$ is the minimum closed set containing $E$.

A subset $E$ of a metric space $X$ is said to be dense in $X$ if its closure is the entire space $X$:

$$\overline{E} = X.$$  

If $E$ is dense in $X$, then every ball in $X$ contains a point from $E$.

A metric space is called separable if it has a countable dense subset.

**Example 2.9.** a) The metric space of real numbers $\mathbb{R}$ is separable because the set of rational numbers $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

b) The space $\mathbb{C}$ of complex numbers is also separable (cf. Exercise 2.23).

c) A discrete metric space is separable if and only if it is countable (cf. Exercise 2.24).
2.3 Convergence and Completeness

Definition 2.6. A point $x$ in a metric space $X$ is said to be a limit of a sequence of points $(x_n)$ in $X$ if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \quad \text{for all } n > N.$$ 

If $x$ is a limit of the sequence $(x_n)$, we say that $(x_n)$ converges to $x$ and write

$$x_n \to x.$$ 

If a sequence has a limit, it is called convergent. Otherwise, it is called divergent.

It is not difficult to see that $x_n \to x$ if and only if $d(x_n, x) \to 0$.

Example 2.10. Every point $x$ in a metric space $X$ is a limit of the constant sequence: $x_n = x$, for all $n \in \mathbb{N}$.

Example 2.11. Let $X = [0, 1]$ be a subspace of $\mathbb{R}$. The sequence $(1/n)$ converges to $0$ in $X$. On the other hand, the same sequence diverges in the subspace $(0, 1)$ of $\mathbb{R}$ (cf. Exercise 2.34).

Theorem 2.5. A sequence of points in a metric space has at most one limit.

Proof. Suppose $(x_n)$ is a convergent sequence such that $x_n \to x$ and $x_n \to y$. Then

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0.$$ 

Hence, $d(x, y) = 0$, so $x = y$. □

This theorem justifies the notation

$$\lim x_n = x$$

for the limit of a sequence $(x_n)$ that converges to $x$.

Definition 2.7. A sequence of points $(x_n)$ in a metric space $X$ is said to be a Cauchy sequence (or fundamental sequence) if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon, \quad \text{for all } m, n > N.$$ 

Theorem 2.6. Every convergent sequence in a metric space is a Cauchy sequence.

Proof. Suppose $x_n \to x$ in a metric space $X$. Then, for a given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that
By the triangle inequality, we have
\[ d(x, y) \leq d(x, z) + d(z, y) \]
for all \( x, y, z \in X \).

**Theorem 2.7.** Any Cauchy sequence in a metric space is bounded. In particular, any convergent sequence in a metric space is bounded.

**Proof.** Let \((x_n)\) be a Cauchy sequence in a metric space \(X\). Then, for \(\varepsilon = 1\), there is \(N \in \mathbb{N}\) such that
\[ d(x_m, x_n) < \varepsilon \]
for all \(m, n > N\). Hence, \((x_n)\) is a Cauchy sequence. \(\square\)

The converse of this theorem is not true in a general metric space. For instance, the sequence \((1/n)\) in the space \((0, 1)\) (cf. Exercise 2.34) is Cauchy but does not converge to a point in \(X\).

A subset \(E\) of a metric space is said to be **bounded** if there is \(M > 0\) such that \(d(x, y) < M\) for all \(x, y \in E\). A sequence \((x_n)\) of points in a metric space is **bounded** if the set \(\{x_n : n \in \mathbb{N}\}\) is bounded.

**Theorem 2.8.** A subspace \(Y\) of a complete space \(X\) is complete if and only if \(Y\) is a closed subset of \(X\).
Proof. (Necessity.) Suppose that $Y$ is complete and let $y \in X$ be a limit point of $Y$. Then, for every $n \in \mathbb{N}$, there is $y_n \in Y$ such that $d(y_n, y) < \frac{1}{n}$. Because $y_n \to y$, the sequence $(y_n)$ is Cauchy in $Y$. Inasmuch as $Y$ is complete, $y \in Y$. Hence $Y$ is a closed subset of $X$.

(Sufficiency.) Let $Y$ be a closed subset of $X$ (so $\overline{Y} = Y$) and let $(y_n)$ be a Cauchy sequence of points in $Y$. Because this sequence is also a sequence of points in $X$ and $X$ is complete, $(y_n)$ converges to a point $x \in X$. If $x \in Y$, we are done. Otherwise, $x$ is a limit point of $Y$, so $x \in \overline{Y} = Y$. □

An important example of an incomplete space is the metric space $\mathbb{Q}$ with the distance function $d(x, y) = |x - y|$ for $x, y \in \mathbb{Q}$. Of course, this follows from the previous theorem because $\mathbb{Q}$ is not a closed subset of the complete space $\mathbb{R}$. To show that $\mathbb{Q}$ is an incomplete metric space directly, it suffices to produce an example of a divergent Cauchy sequence in this space.

Example 2.13. Consider the sequence of rational numbers

$$s_n = 1 - \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n+1} \frac{n}{n!}, \quad n \in \mathbb{N}.$$ 

First, we show that $(s_n)$ is a Cauchy sequence in $\mathbb{Q}$. For $m > n$, we have

$$|s_m - s_n| = \left|(-1)^{n+2} \frac{1}{(n+1)!} + \cdots + (-1)^{m+1} \frac{m}{m!}\right| \leq \frac{1}{2n-1} + \cdots + \frac{1}{2m-2} < \frac{1}{2}.$$ 

(We use the inequality $n! > 2^{n-2}$ for all $n \in \mathbb{N}$, and the sum of a geometric series.) It follows that $(s_n)$ is Cauchy.

Suppose that $(s_n)$ converges to a rational number $r$, that is,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = r.$$ 

Because the series in the left hand side satisfies the Alternating Series Test conditions, we have the following estimate for the remainder (cf. Exercise 2.35):

$$0 < |r - s_n| < \frac{1}{(n+1)!}, \quad \text{for } n \in \mathbb{N}.$$ 

Let $r = \frac{p}{q} = \frac{p(q-1)!}{q!}$ and $n = q$. From the above inequalities, we have

$$0 < \left|\frac{p(q-1)!}{q!} - \sum_{k=1}^{q} \frac{(-1)^{k+1}}{k!}\right| < \frac{1}{(q+1)!}.$$ 

By multiplying all sides of these inequalities by $q!$, we obtain a contradiction.
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\[ 0 < \left| p(q - 1)! - \sum_{k=1}^{q} \frac{(-1)^{k+1} q!}{k!} \right| < \frac{1}{q + 1}, \]

because the number in the middle is an integer. It follows that the Cauchy sequence \((s_n)\) diverges in \(\mathbb{Q}\).

The diameter of a bounded set \(A\) in a metric space \((X, d)\) is defined as

\[ \text{diam}(A) = \sup \{d(a, b) : a, b \in A \}. \]

A nested family \(A_1 \supseteq A_2 \supseteq \cdots\) of nonempty subsets of \(X\) is said to be contracting if

\[ \text{diam}(A_n) \to 0. \]

The following property of complete metric spaces is useful in applications.

**Theorem 2.9. (The Cantor Intersection Property)** If a metric space \(X\) is complete then whenever \(A_1 \supseteq A_2 \supseteq \cdots\) is a contracting sequence of nonempty closed subsets of \(X\), then there is \(x \in X\) for which

\[ \bigcap_{k=1}^{\infty} A_k = \{x\}. \]

**Proof.** Let \(r_n = \text{diam}(A_n)\), for \(n \in \mathbb{N}\). The sequence \((r_n)\) is decreasing and converges to zero. For each \(n \in \mathbb{N}\), choose \(x_n \in A_n\). For a given \(\varepsilon > 0\), there is \(N \in \mathbb{N}\) such that \(r_n < \varepsilon\) for all \(n > N\). Hence, for all \(m, n > N\), \(d(x_m, x_n) < \varepsilon\), so \((x_n)\) is Cauchy. Therefore, there is \(x \in X\) such that \(x_n \to x\).

Inasmuch as the sets \(A_n\) are closed, \(x \in A_n\) for all \(n \in \mathbb{N}\), so \(x \in \bigcap_{k=1}^{\infty} A_k\).

The claim follows because \(r_n \to 0\). \(\Box\)

**Example 2.14.** We show that the set of real numbers \(\mathbb{R}\) is uncountable. The proof is by contradiction, so we suppose that there is a sequence of real numbers

\[ x_1, x_2, \ldots, x_n, \ldots \]

such that every real number is a term of this sequence.

It is clear that there is a closed bounded interval \([a_1, b_1]\) of length

\[ b_1 - a_1 = \frac{1}{3} \]

such that \(x_1 \notin [a_1, b_1]\). We divide this interval into three closed subintervals of equal length. Again, it is clear that there is at least one of these subintervals, say \([a_2, b_2]\), that does not contain \(x_2\) (and does not contain \(x_1\) either). By using recursion, we construct a nested family of closed intervals

\[ \{ [a_n, b_n] : n \in \mathbb{N} \} \]
such that, for all $n \in \mathbb{N}$,

$$b_n - a_n = \frac{1}{3^n},$$

and $[a_n, b_n]$ does not contain numbers $x_1, \ldots, x_n$. By Theorem 2.9, there is a real number $a$ that belongs to all intervals $[a_n, b_n]$ and hence is not a term of the sequence $(x_n)$.

### 2.4 Mappings

Let $X$ and $Y$ be sets. Recall that a function $f : X \to Y$ is a subset $f \subseteq X \times Y$ satisfying the following condition: for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. Then we write $y = f(x)$ and call $X$ the domain of the function $f$ and the set $Y$ its codomain.

In functional analysis, we often use different names instead of “function” such as “mapping”, “transformation”, “operator” among other variations. Some notations are also abbreviated to make formulae more readable.

The following definition illustrates some of these conventions. It is clear that this definition generalizes the concept of continuity in elementary analysis.

**Definition 2.9.** Let $(X, d)$ and $(Y, d')$ be metric spaces. A mapping (that is, a function) $T : X \to Y$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \text{implies} \quad d'(Tx, Tx_0) < \varepsilon, \quad \text{for all } x \in X.$$

The mapping $T$ is said to be continuous (on $X$) if it is continuous at every point of $X$.

Note that we write $Tx$ instead of the usual notation $T(x)$ for the value of a function at $x$.

The next two theorems give equivalent forms of this definition.

**Theorem 2.10.** A mapping $T : X \to Y$ is continuous at $x_0 \in X$ if and only if $Tx_n \to Tx_0$ for every sequence $(x_n)$ in $X$ such that $x_n \to x_0$.

**Proof.** (Necessity.) Let $T : X \to Y$ be a continuous mapping and $x_n \to x_0$ in the space $X$. For $\varepsilon > 0$ there is $\delta > 0$ such that $d'(Tx, Tx_0) < \varepsilon$ whenever $d(x, x_0) < \delta$. Because $x_n \to x_0$, there is $N \in \mathbb{N}$ such that $d(x_n, x_0) < \delta$ for $n > N$. Hence, $d'(Tx_n, Tx_0) < \varepsilon$ for $n > N$. It follows that $Tx_n \to Tx_0$ in the space $Y$.

(Sufficiency.) Suppose that $T$ is not continuous at $x_0$. Then there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there is $x_n \in X$ such that $d(x_n, x_0) < 1/n$ and $d'(Tx_n, Tx_0) \geq \varepsilon$. It is clear that $(Tx_n)$ does not converge to $Tx_0$ in $Y$. 

whereas \( x_n \to x_0 \) in \( X \). This contradiction shows that \( T \) is continuous at \( x_0 \).

\[ \square \]

**Theorem 2.11.** A mapping \( T : X \to Y \) is continuous on \( X \) if and only if the inverse image of any open set in \( Y \) is an open set in \( X \).

**Proof.** (Necessity.) Let \( T : X \to Y \) be a continuous mapping and \( U \) be an open set in \( Y \). For \( x_0 \in T^{-1}(U) \) choose \( \varepsilon > 0 \) such that \( B(T x_0, \varepsilon) \subseteq U \). This is possible because \( T x_0 \in U \) and \( U \) is an open set. Because \( T \) is continuous, there is \( \delta > 0 \) such that \( d(x, x_0) < \delta \) implies \( T x \in B(T x_0, \varepsilon) \). It follows that \( B(x_0, \delta) \subseteq T^{-1}(U) \), so \( T^{-1}(U) \) is an open subset of \( X \).

(Sufficiency.) Suppose that the inverse image of any open set in \( Y \) is an open set in \( X \). Let \( x_0 \) be a point in \( X \) and \( \varepsilon > 0 \). Let \( V \subseteq X \) be the inverse image of the open ball \( B(T x_0, \varepsilon) \). Because \( V \) is an open set and \( x_0 \in V \), there is \( \delta > 0 \) such that \( B(x_0, \delta) \subseteq V \), that is, \( d(x, x_0) < \delta \) implies \( d'(T x, T x_0) < \varepsilon \) for all \( x \in X \). It follows that \( T \) is continuous at \( x_0 \).

\[ \square \]

An example of a continuous function is the distance function on a metric space. This function is continuous in both its variables. Here is the precise formulation of this property.

**Theorem 2.12.** Let \( (X, d) \) be a metric space. If \( x_n \to x \) and \( y_n \to y \) in \( X \), then

\[ d(x_n, y_n) \to d(x, y), \]

or, equivalently,

\[ d(\lim x_n, \lim y_n) = \lim d(x_n, y_n), \]

provided that both limits on the left hand side exist.

**Proof.** The claim of the theorem follows from the “quadrilateral inequality”:

\[ |d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \]

(cf. Exercise 2.3).

\[ \square \]

Another example of a continuous mapping is an isometry.

**Definition 2.10.** Let \( (X, d) \) and \( (Y, d') \) be metric spaces. A mapping \( T \) of \( X \) into \( Y \) is said to be an isometry if it preserves distances, that is,

\[ d'(T x, T y) = d(x, y), \quad \text{for all } x, y \in X \]

The space \( X \) is said to be isometric with the space \( Y \) if there is a bijective isometry from \( X \) onto \( Y \). Then the spaces \( X \) and \( Y \) are called isometric.

It is not difficult to see that an isometry is indeed a continuous mapping.

The claim of the next theorem is an important property of isometric spaces.
Theorem 2.13. Let \((X, d)\) and \((X', d')\) be complete metric spaces. If \(U\) and \(U'\) are dense subsets of \(X\) and \(X'\), respectively, and \(T\) is an isometry from \(U\) onto \(U'\), then there exists an isometry \(T'\) from \(X\) onto \(X'\) such that \(T = T'|_U\), that is, \(T'\) is an extension of \(T\) to \(X\).

In other words, if \(U\) and \(U'\) are isometric dense subsets of complete metric spaces \(X\) and \(X'\), respectively, then the spaces \(X\) and \(X'\) are themselves isometric.

Proof. Let \(x\) be a point in \(X\). Because \(U\) is dense in \(X\), there is a sequence \((u_n)\) of points of \(U\) that converges to \(x\). Suppose that \((v_n)\) is another sequence with the same properties. Since convergent sequence are Cauchy and \(T\) preserves distances, the sequences \((T u_n)\) and \((T v_n)\) are also Cauchy. They are convergent because \(X'\) is complete. By Theorem 2.12,

\[
d'(\lim T u_n, \lim T v_n) = \lim d'(T u_n, T v_n) = \lim d(u_n, v_n) = d(\lim u_n, \lim v_n) = d(x, x) = 0.
\]

Hence, \(\lim T u_n = \lim T v_n\) for any two sequences in \(U\) that converge to \(x\). It follows that \(T' x = \lim T u_n\) is a well-defined function from \(X\) into \(X'\).

Now we show that \(T'\) preserves distances. Let \((u_n)\) and \((v_n)\) be sequences of points in \(U\) that converge to points \(x\) and \(y\) in \(X\), respectively. We have, again by Theorem 2.12,

\[
d'(T' x, T' y) = d'(\lim T u_n, \lim T v_n) = \lim d'(T u_n, T v_n) = \lim d(u_n, v_n) = d(\lim u_n, \lim v_n) = d(x, y).
\]

Hence, the mapping \(T'\) is an isometry.

It remains to show that \(T'\) is a surjective mapping. Let \(x'\) be a point in \(X'\). Because \(U'\) is dense in \(X'\), there is a sequence \((u'_n)\) of points of \(U'\) that converges to \(x'\). Because this sequence converges, it is Cauchy. Since \(T^{-1}\) is an isometry, the sequence \(T^{-1} u'_n\) is also a Cauchy sequence. Inasmuch as \(X\) is a complete space, this sequence converges to some point \(x\) in \(X\), that is, \(u_n \to x\), where \(u_n = T^{-1} u'_n\). Hence,

\[
T' x = \lim T u_n = \lim u'_n = x',
\]

and the result follows. \(\square\)

2.5 Completion of a Metric Space

The completeness property of the field of real numbers \(\mathbb{R}\) is crucial in the proofs of many theorems of real analysis. Complete metric spaces play an important role in many constructions of functional analysis. However, some-
times one has to deal with incomplete metric spaces. In this section, we
describe a remarkable construction that makes it possible to create a unique
complete space from an incomplete one. We begin with a definition.

**Definition 2.11.** A completion of a metric space \((X, d)\) is a metric space
\((\tilde{X}, \tilde{d})\) with the following properties:

a) \(\tilde{X}\) is a complete space, and

b) \((X, d)\) is isometric with a dense subset of \((\tilde{X}, \tilde{d})\).

It is known from real analysis that the space of real numbers \(\mathbb{R}\) is a com-
pletion of its subspace of rational numbers \(\mathbb{Q}\).

**Theorem 2.14.** For a metric space \((X, d)\) there exists its completion
\((\tilde{X}, \tilde{d})\).

**Proof.** First, we construct the space \((\tilde{X}, \tilde{d})\). Let \(X'\) be the set of Cauchy
sequences of elements of \(X\). We define the function \(d'(\langle x_n \rangle, \langle y_n \rangle)\) on \(X'\) by

\[ d'(\langle x_n \rangle, \langle y_n \rangle) = \lim_{n \to \infty} d(x_n, y_n). \]

This function is well-defined. Indeed, we have

\[ |d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n), \quad \text{for all } m, n \in \mathbb{N}, \]

by the “quadrilateral inequality” (cf. Exercise 2.3). Because \((x_n)\) and \((y_n)\)
are Cauchy sequences in \(X\), the sequence of real numbers \(d(x_n, y_n)\) is Cauchy
and therefore convergent.

It is not difficult to verify that the function \(d'\) is a pseudometric (cf. Ex-
ercise 2.4) on \(X'\). Let \(\tilde{X} = X' / \sim\) be the quotient set of \(X'\) with respect to
the equivalence relation on \(X'\) defined by

\[ \langle x_n \rangle \sim \langle y_n \rangle \quad \text{if and only if} \quad d'(\langle x_n \rangle, \langle y_n \rangle) = 0 \]

According to Exercise 2.4, the set \(\tilde{X}\) equipped with the distance function

\[ \tilde{d}(\langle x_n \rangle, \langle y_n \rangle) = d'(\langle x_n \rangle, \langle y_n \rangle) \]

is a metric space.

We proceed with construction of an isometry \(T : X \to \tilde{X}\). With each
\(x \in X\) we associate the class \([\langle x \rangle]\) of the constant sequence \(\langle x \rangle\) and claim
that the mapping given by \(Tx = [\langle x \rangle]\) is an isometry from \(X\) onto \(T(X) \subseteq \tilde{X}\).

Indeed, for constant sequences \(\langle x \rangle\) and \(\langle y \rangle\), we have

\[ \tilde{d}(\langle x \rangle, \langle y \rangle) = d'(\langle x \rangle, \langle y \rangle) = d(x, y), \]

by definitions of functions \(d\) and \(d'\).

Now we show that \(T(X)\) is dense in \(\tilde{X}\). For this we need to show that
every open ball in \(\tilde{X}\) contains a point of \(T(X)\). Let \([\langle x_n \rangle] \in \tilde{X}\). Because \((x_n)\)
is a Cauchy sequence in \(X\), for every \(\varepsilon > 0\) there is \(N \in \mathbb{N}\) such that
Consider the constant sequence \((x_N) = (x_N, x_N, \ldots)\) in \(X\), so \([x_N] \in T(X)\).

We have
\[
d([x_n], [x_N]) = d'(x_n, x_N) = \lim d(x_n, x_N) \leq \frac{\varepsilon}{2} < \varepsilon.
\]

Hence, the ball \(B([x_N], \varepsilon)\) contains the point \([x_N]\) of \(T(X)\).

It remains to show that \((\tilde{X}, \tilde{d})\) is a complete space. Let \((\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, \ldots)\) be a Cauchy sequence in \(\tilde{X}\). Because \(T(X)\) is dense in \(\tilde{X}\), for every \(n \in \mathbb{N}\), there is a constant sequence \((\tilde{z}^{(n)})\) = \((\tilde{z}_n, \tilde{z}_n, \ldots)\) in \(X\) such that
\[
\tilde{d}(\tilde{x}_n, \tilde{z}_n) < \frac{1}{n},
\]
where \(\tilde{z}_n = [(\tilde{z}^{(n)})] \in T(X)\) is the class of the constant sequence \((\tilde{z}^{(n)})\). We have
\[
\tilde{d}(\tilde{z}_m, \tilde{z}_n) \leq \tilde{d}(\tilde{z}_m, \tilde{x}_m) + \tilde{d}(\tilde{x}_m, \tilde{x}_n) + \tilde{d}(\tilde{x}_n, \tilde{z}_n) < \frac{1}{m} + \tilde{d}(\tilde{x}_m, \tilde{x}_n) + \frac{1}{n}
\]
and the right side can be made smaller than any given positive number for sufficiently large \(m\) and \(n\) because \((\tilde{x}_n)\) is a Cauchy sequence. Hence, \((\tilde{z}_n)\) is Cauchy. Because \(T\tilde{z}_n = [(\tilde{z}^{(n)})] = \tilde{z}_n\) and \(T\) is an isometry of \(X\) onto \(T(X)\), the sequence \((\tilde{z}_n) = (\tilde{z}_1, \tilde{z}_2, \cdots)\) is a Cauchy sequence in \(X\). Let \(\tilde{x} = [(\tilde{z}_n)]\). We show that \(\tilde{x}_n \to \tilde{x}\), which establishes completeness of \(\tilde{X}\).

We have
\[
\tilde{d}(\tilde{x}_n, \tilde{x}) \leq \tilde{d}(\tilde{x}_n, \tilde{z}_n) + \tilde{d}(\tilde{z}_n, \tilde{x}) < \frac{1}{n} + \tilde{d}(\tilde{z}_n, \tilde{x}) = \frac{1}{n} + d'((z^{(n)}), (z_n))
\]
By using the definition of \(d'\), we obtain
\[
\tilde{d}(\tilde{x}_n, \tilde{x}) < \frac{1}{n} + \lim_{m \to \infty} d(z_n, z_m).
\]
Because \((z_n)\) is a Cauchy sequence, the right side of the above inequality can be made smaller than any given positive number for sufficiently large \(n\), implying that \(\tilde{x}_n \to \tilde{x}\).

The next theorem asserts that in some precise sense there is only one completion of a metric space.

**Theorem 2.15.** If \((\tilde{X}, \tilde{d})\) and \((\tilde{X}', \tilde{d}')\) are two completions of a metric space \((X, d)\), then the spaces \(\tilde{X}\) and \(\tilde{X}'\) are isometric.

*Proof.* Let \(T\) and \(T'\) be isometries of \(X\) onto dense sets \(T(X)\) and \(T'(X)\) in spaces \(\tilde{X}\) and \(\tilde{X}'\), respectively. Then \(T' \circ T^{-1}\) is an isometry of \(T(X)\) onto \(T'(X)\). The claim of the theorem follows from Theorem 2.13. \(\square\)
2.6 The Baire Category Theorem

The result that we establish in this section—the Baire Category Theorem—has powerful applications in functional analysis and elsewhere in mathematics. We begin with a simple lemma.

**Lemma 2.1.** A nonempty open set $U$ in a metric space $X$ contains the closure of an open ball.

*Proof.* Let $B(x, r)$ be an open ball in $U$. Clearly, $B(x, r/2) \subseteq B(x, r)$. Let $y$ be a limit point of $B(x, r/2)$. There is $z \in B(x, r/2)$ such that $d(z, y) < r/2$. Hence,

$$d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2 = r,$$

so $y \in B(x, r)$. It follows that $\overline{B(x, r/2)} \subseteq B(x, r) \subseteq U$. □

Next we prove two forms of the main result. Both are known as the Baire Category Theorem.

**Theorem 2.16.** Let $X_1, X_2, \ldots$ be a sequence of dense open sets in a complete metric space $(X, d)$. Then

$$\bigcap_{n=1}^{\infty} X_n \neq \emptyset.$$

*Proof.* By Lemma 2.1, there is an open ball $B(x_1, \varepsilon_1)$ such that

$$\overline{B(x_1, \varepsilon_1)} \subseteq X_1.$$

Because $X_2$ is dense in $X$, the intersection $B(x_1, \varepsilon_1) \cap X_2$ is a nonempty open set and therefore, by Lemma 2.1, there is an open ball $B(x_2, \varepsilon_2)$ such that

$$B(x_1, \varepsilon_1) \supseteq B(x_2, \varepsilon_2) \quad \text{and} \quad \overline{B(x_2, \varepsilon_2)} \subseteq X_2.$$

By recursion, we construct a nested sequence of open balls such that

$$B(x_1, \varepsilon_1) \supseteq B(x_2, \varepsilon_2) \supseteq \cdots \supseteq B(x_n, \varepsilon_n) \supseteq \cdots$$

and

$$\overline{B(x_n, \varepsilon_n)} \subseteq X_n, \quad \text{for all } n \in \mathbb{N}.$$

It is clear that we may assume that $\varepsilon_n \to 0$. By Theorem 2.9,

$$\bigcap_{n=1}^{\infty} X_n \supseteq \bigcap_{n=1}^{\infty} \overline{B(x_n, \varepsilon_n)} \neq \emptyset$$

(cf. Exercise 2.37), which is the required result. □
2.7 Compactness

**Theorem 2.17.** Let \((X, d)\) be a complete metric space. If \(X = \bigcup_{n=1}^{\infty} F_n\) where \(F_1, F_2, \ldots\) are closed sets, then at least one of these sets contains an open ball.

**Proof.** Suppose that none of the closed sets \(F_1, F_2, \ldots\) contains an open ball. Then each of the open sets \(X \setminus F_1, X \setminus F_2, \ldots\) is dense in \(X\) and

\[
\bigcap_{n=1}^{\infty} (X \setminus F_n) = X \setminus \bigcup_{n=1}^{\infty} F_n = \emptyset,
\]

which contradicts the result of Theorem 2.16. \(\square\)

**Example 2.15.** The set of real numbers \(\mathbb{R}\) is not countable (cf. Example 2.14). Indeed, suppose that it is countable. Since \(\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}\) and each set \(\{x\}\) is closed, we have a contradiction with Theorem 2.17.

2.7 Compactness

**Definition 2.12.** A metric space \(X\) is said to be **compact** if every sequence in \(X\) has a convergent subsequence. A subset \(Y\) of \(X\) is called **compact** if it is a compact subspace of \(X\).

Hence a subset \(Y\) of a metric space \(X\) is compact if any sequence in this subset has a subsequence that converges to a point in \(Y\).

**Example 2.16.** In real analysis, the Bolzano-Weierstrass Theorem asserts that a subset of \(\mathbb{R}^n\) is compact if and only if it is closed and bounded.

**Theorem 2.18.** A compact set in a metric space is closed and bounded.

**Proof.** Let \(E\) be a compact set in a metric space \(X\). If \(x\) is a point in the closure of \(E\), then there is a sequence \((x_n)\) of points in \(E\) that converges to \(x\) in the space \(X\). Because \(E\) is compact, \(x\) belongs to \(E\).

Suppose that the set \(E\) is not bounded. Then, for every \(n \in \mathbb{N}\), there is a point \(y_n \in E\) such that \(d(y_n, a) > n\), where \(a\) is a fixed point in \(X\). Because every convergent sequence of points in \(X\) is bounded, the sequence \((y_n)\) cannot contain a convergent subsequence. This contradicts our assumption that \(E\) is compact. \(\square\)

The converse of this theorem is false for general metric spaces as the following example demonstrates.

**Example 2.17.** Let \(X\) be an infinite set endowed with the discrete metric

\[d(x, y) = 1, \quad \text{for all } x \neq y \text{ in } X.\]
The set $X$ is closed and bounded. Clearly, a sequence $(x_n)$ with all distinct terms does not contain a convergent subsequence. (Any subsequence of this sequence is not Cauchy.) Hence, $X$ is not compact.

Let $Y$ be a subspace of a metric space $X$. A family $\{U_i\}_{i \in J}$ of open sets in $X$ is said to be an open covering of $Y$ if $Y \subseteq \bigcup_{i \in J} U_i$. If $J' \subseteq J$, then the family $\{U_i\}_{i \in J'}$ is called an open subcovering of $\{U_i\}_{i \in J}$ if $Y \subseteq \bigcup_{i \in J'} U_i$.

**Lemma 2.2.** Let $\{U_i\}_{i \in J}$ be an open covering of a compact space $X$. There exists $r > 0$ such that for every $x \in X$ the open ball $B(x, r)$ is contained in $U_i$ for some $i \in J$.

**Proof.** Suppose to the contrary that for every $n \in \mathbb{N}$ there is $x_n \in X$ such that $B(x_n, 1/n) \not\subseteq U_i$ for all $i \in J$. Inasmuch as $X$ is compact, there is a subsequence $(x_{n_k})$ of the sequence $(x_n)$ converging to some $x \in X$. The point $x$ belongs to one of the sets $U_i$, say, $x \in U_{i_0}$. Because $U_{i_0}$ is open, there is $m$ such that $B(x, 1/m) \subseteq U_{i_0}$. Since $x_{n_k} \to x$, there is $n_0 \geq 2m$ such that $x_{n_0} \in B(x, 1/2m)$. We have (cf. Exercise 2.13.b)

$$B\left(x_{n_0}, \frac{1}{n_0}\right) \subseteq B\left(x_{n_0}, \frac{1}{2m}\right) \subseteq B\left(x, \frac{1}{m}\right) \subseteq U_{i_0},$$

which contradicts our assumption that $B(x_n, 1/n) \not\subseteq U_i$ for all $n \in \mathbb{N}$ and $i \in J$. \hfill $\square$

**Theorem 2.19. (Borel-Lebesgue)** A metric space $X$ is compact if and only if any open covering $\{U_i\}_{i \in J}$ of $X$ contains a finite subcovering.

**Proof.** (Necessity.) Let $X$ be a compact space and $\{U_i\}_{i \in J}$ an open covering of $X$. By Lemma 2.2, there is $r > 0$ such that, for every $x \in X$, $B(x, r) \subseteq U_i$ for some $i \in J$. It suffices to prove that $X$ can be covered by a finite number of balls $B(x, r)$. If $B(x_1, r) = X$ for some $x_1 \in X$, we are done. Otherwise, choose $x_2 \in X \setminus B(x_1, r)$. If $B(x_1, r) \cup B(x_2, r) = X$, the proof is over. If by continuing this process we obtain $X$ on some step, the claim is proven. Otherwise, there exists a sequence $(x_n)$ of points of $X$ such that

$$x_{n+1} \notin B(x_1, r) \cup \cdots \cup B(x_n, r)$$

for every $n \in \mathbb{N}$. It is clear that $d(x_m, x_n) \geq r$ for all $m, n \in \mathbb{N}$. It follows that any subsequence of $(x_n)$ is not Cauchy and therefore is not convergent. This contradicts our assumption that $X$ is a compact space. Hence, $X$ can be covered by a finite number of sets $U_i$.

(Sufficiency.) Let $X$ be a metric space such that any open covering of $X$ contains a finite subcovering and let $(x_n)$ be a sequence of points in $X$. Suppose that $(x_n)$ does not have a convergent subsequence. Then for every point $x \in X$ there is an open ball $B(x, r_x)$ that contains no points of the sequence $(x_n)$ with $n \in \mathbb{N}$. As each $x_n$ lies in some $U_i$, we can find an open ball $B(x_n, r_{x_n})$ such that $B(x_n, r_{x_n}) \subseteq U_i$. But then we have $B(x_n, r_{x_n}) \subseteq B(x, r_x)$ for all $n$, contradicting our assumption. Therefore, $(x_n)$ must converge to some point $x$. Therefore, $X$ is compact.
(x_n) except possibly x itself (cf. Exercise 2.27). The family \( \{ B(x, r_x) \}_{x \in X} \) is an open covering of X and therefore contains a finite subcovering. Hence,

\[
X = B(a_1, r_1) \cup \cdots \cup B(a_n, r_n),
\]

for a finite set \( A = \{ a_1, \ldots, a_n \} \) in X. By the choice of open balls \( B(x, r_x) \), \( x_k \in A \) for all \( k \in \mathbb{N} \), which contradicts our assumption that \( (x_n) \) does not have a convergent subsequence (cf. Exercise 2.26).

\[ \square \]

**Theorem 2.20.** The image of a compact set under a continuous mapping is compact.

*Proof.* Let \( T : X \to Y \) be a continuous mapping and \( A \) a compact subset of \( X \). If \( \{ U_i \}_{i \in I} \) is an open covering of \( T(A) \), then, by Theorem 2.19, \( \{ T^{-1}(U_i) \}_{i \in I} \) is an open covering of \( A \). Inasmuch as \( A \) is compact, there is a subcovering \( \{ T^{-1}(U_i) \}_{i \in J'} \) of \( A \) where \( J' \) is a finite subset of \( J \). Then \( \{ U_i \}_{i \in J'} \) is a finite subfamily of \( \{ U_i \}_{i \in I} \) which covers \( T(A) \). Hence, \( T(A) \) is a compact subset of the space \( Y \). \[ \square \]

**Theorem 2.21.** If \( X \) is a compact space, then every continuous function \( f : X \to \mathbb{R} \) is bounded and attains its maximum value.

*Proof.* By Theorem 2.20, the set \( A = f(X) \subseteq \mathbb{R} \) is compact. If \( A \) has no largest element, then the family

\[
\{ (-\infty, a) : a \in A \}
\]

forms an open covering of \( A \). Inasmuch as \( A \) is compact, some finite subfamily

\[
\{ (-\infty, a_1), \ldots, (-\infty, a_n) \}
\]

covers \( A \). If \( a_k = \max\{a_1, \ldots, a_n\} \), then for all \( 1 \leq i \leq n \), \( a_k \notin (-\infty, a_i) \), contrary to the fact that these intervals cover \( A \). \[ \square \]

### 2.8 Topological Spaces

Many problems in functional analysis require the consideration of topological spaces, a more general structure than that of a metric space.

**Definition 2.13.** A topological space is a pair \((X, \mathcal{T})\) where \( X \) is a nonempty set and \( \mathcal{T} \) is a family of subsets of \( X \), called open sets, possessing the following properties:

a) The union of any collection of open sets is open;
b) The intersection of any finite collection of open sets is open;
c) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

Elements of the set $X$ are called points of the topological space $(X, \mathcal{T})$ and the family $\mathcal{T}$ is called a topology on $X$. For a point $x \in X$, a neighborhood of $x$ is a set containing an open set which contains $x$.

It is worth noting that property c) of the definition follows from properties a) and b). Namely, the union of the empty collection of open sets is the empty set $\emptyset$, and the intersection of the empty collection is the set $X$ itself.

As in the case of metric spaces, it is a custom to omit reference to the topology $\mathcal{T}$ in the notation $(X, \mathcal{T})$ and write “a topological space $X$” instead of “a topological space $(X, \mathcal{T})$”.

Let $(X, \mathcal{T})$ be a topological space. If $Y$ is a subset of $X$, then the family $\mathcal{T}' = \{Y \cap U : U \in \mathcal{T}\}$ is a topology on $Y$ (cf. Exercise 2.51). This topology is called the subspace topology.

An important example of a topological space is a metric space. Let $(X, d)$ be a metric space and $\mathcal{T}$ be the family of all open sets in $X$. Note that $\mathcal{T}$ contains the empty set $\emptyset$ and the set $X$ itself. By Theorem 2.1, the pair $(X, \mathcal{T})$ is a topological space. In this case, the topology $\mathcal{T}$ is called the metric topology induced by the metric $d$.

**Example 2.18.** Let $X$ be a nonempty set. Define $\mathcal{T} = \{\emptyset, X\}$. It is clear that $\mathcal{T}$ is a topology on $X$. It is called the trivial topology on $X$.

**Example 2.19.** Let $X$ be a nonempty set and $\mathcal{P}$ be the family of all subsets of $X$. Hence, $\mathcal{P}$ is the power set $2^X$. This topology is called the discrete topology on $X$. The discrete topology is induced by the discrete metric.

A topological space is said to be Hausdorff if any two distinct points in the space have disjoint neighborhoods.

**Example 2.20.**

a) Any metric space $(X, d)$ is Hausdorff. Indeed, if $x$ and $y$ are distinct points in $X$, then $B(x, d(x, y)/2) \cap B(y, d(x, y)/2) = \emptyset$.

b) Let $X = \{a, b\}$ be a 2-element set and $\mathcal{T} = \{\emptyset, \{a\}, X\}$. It is clear that $\mathcal{T}$ is a topology on $X$ and that the space $(X, \mathcal{T})$ is not Hausdorff.

**Example 2.21.** Let $(X, d)$ be a pseudometric space which is not a metric space, so $d(x, y) = 0$ for some distinct points $x$ and $y$ in $X$. Open balls and open sets in this space are defined exactly as in the case of metric spaces (cf. Definitions 2.3 and 2.4). It is not difficult to show that the space $X$ is not Hausdorff (cf. Exercise 2.50).

A base (or basis) of the topology $\mathcal{T}$ on a nonempty set $X$ is any family $\mathcal{B}$ of open subsets of $X$ such that every open subset of $X$ is the union of sets belonging to $\mathcal{B}$. If $\mathcal{B}$ is a base of the topology $\mathcal{T}$, we say that $\mathcal{B}$ generates $\mathcal{T}$.
Bases are useful because many topologies are most easily defined in terms of bases generating them. The following theorem (cf. Exercise 2.52) gives necessary and sufficient conditions for a family of subsets to generate a topology.

**Theorem 2.22.** Let \( \mathcal{B} \) be a family of subsets of a nonempty set \( X \). Then \( \mathcal{B} \) is a base of a topology on \( X \) if and only if

a) \( X = \bigcup_{B \in \mathcal{B}} B \),

b) if \( B_1, B_2 \in \mathcal{B} \) and \( x \in B_1 \cap B_2 \), then there is a set \( B \) in \( \mathcal{B} \) such that \( x \in B \subseteq B_1 \cap B_2 \).

The unique topology that has \( \mathcal{B} \) as its base consists of the unions of subfamilies of \( \mathcal{B} \).

**Example 2.22.** It is known from real analysis that any open subset of \( \mathbb{R} \) is the union of (possibly empty) family of open intervals. Hence, the open intervals form a base of the metric topology on \( \mathbb{R} \). In fact, in any metric space \( X \), the collection of open balls is a base of the metric topology on \( X \).

**Definition 2.14.** Let \( X \) and \( Y \) be topological spaces. A mapping \( T : X \rightarrow Y \) is said to be continuous at a point \( x_0 \in X \) if to every neighborhood \( U \) of the point \( T x_0 \) there corresponds a neighborhood \( V \) of point \( x_0 \) such that \( T(V) \subseteq U \). The mapping \( T \) is said to be continuous if it is continuous at every point of \( X \).

**Definition 2.15.** Topological spaces \( X \) and \( Y \) are said to be homeomorphic if there is a bijection \( T : X \rightarrow Y \) such that both mappings \( T \) and \( T^{-1} \) are continuous. A topological space is said to be metrizable if it is homeomorphic to a metric space.

If two metric spaces are isometric, then they are homeomorphic as topological spaces. Note that the converse is not true (cf. Exercise 2.53).

We have the following analog of Theorem 2.11.

**Theorem 2.23.** Let \( X \) and \( Y \) be topological spaces and \( T \) a mapping from \( X \) into \( Y \). Then \( T \) is continuous if and only if the inverse image under \( T \) of every open set in \( Y \) is an open set in \( X \).

**Proof.** (Necessity.) Let \( T : X \rightarrow Y \) be a continuous mapping and \( U \) an open set in \( Y \). Then \( U \) is a neighborhood of every point \( T x \) for \( x \in T^{-1}(U) \). Because \( T \) is continuous, every \( x \in T^{-1}(U) \) has an open neighborhood \( V \) such that \( T(V) \subseteq U \), so \( V \subseteq T^{-1}(U) \). Hence, \( T^{-1}(U) \) is an open set.

(Sufficiency.) Suppose that the inverse image of every open set in \( Y \) is an open set in \( X \). For \( x \in X \), a neighborhood of \( T x \) contains an open subset \( U \) containing \( T x \). The set \( V = T^{-1}(U) \) is an open neighborhood of \( x \) such that \( T(V) \subseteq U \). Hence, \( T \) is continuous at \( x \). Because \( x \) is an arbitrary point of \( X \), the mapping \( T \) is continuous. \( \square \)
Definition 2.16. If \((X_1, \mathcal{T}_1)\) and \((X_2, \mathcal{T}_2)\) are topological spaces, a topology on the Cartesian product \(X_1 \times X_2\) is defined by taking as a base the collection of all sets of the form \(U_1 \times U_2\), where \(U_1 \in \mathcal{T}_1\) and \(U_2 \in \mathcal{T}_2\). It can be verified (cf. Exercise 2.54) that this collection is indeed a base of a topology. This unique topology is called the product topology on \(X_1 \times X_2\).

The result of Theorem 2.19 motivates the following definition.

Definition 2.17. A topological space \((X, \mathcal{T})\) is said to be compact if every covering of \(X\) by open sets (that is, an open covering) contains a finite subcovering. A subset \(Y\) of \(X\) is said to be compact if \(Y\) is a compact space in the subspace topology.

The proofs of the next two theorems can be taken verbatim from the proofs of Theorems 2.20 and 2.21.

Theorem 2.24. Let \(T : X \to X'\) be a continuous mapping of a topological space \((X, \mathcal{T})\) into a topological space \((X', \mathcal{T}')\). Then the image \(T(Y)\) of a compact set \(Y\) in \(X\) is a compact set in \(X'\).

Theorem 2.25. If \(X\) is a compact space, then every continuous function \(f : X \to \mathbb{R}\) is bounded and attains its maximum value.

Notes

Axioms (M1)–(M3) are motivated by the classical Euclidean geometry where, in particular, it is proven that each side of a triangle is smaller than the sum of the other two sides, and each side is greater than the difference of the other two sides (see, for instance, Kiselev, 2014, pp. 38–39).

There is a loose connection between the concept of a limit and the one of a limit point of a subset. Let \(E\) be a nonempty subset of a metric space and \(x\) a limit point of \(E\). For every \(n \in \mathbb{N}\), there is a point \(x_n \in E\) (distinct from \(x\)) such that \(d(x_n, x) < 1/n\), so \(x_n \to x\). Hence, a limit point of the set \(E\) is the limit of a sequence of points in \(E\). The converse is not true. For instance, if \(E = X\) where \(X\) is a discrete space, then \(E\) has no limit points, whereas any point of \(E\) is a limit of a constant sequence.

Exercises

2.1. Prove that properties M1–M3 of the metric \(d\) are equivalent to the following conditions:

\(d(x, y) = 0\) if and only if \(x = y\),
(2) \[ d(x, y) \leq d(x, z) + d(y, z), \]
for all \( x, y, z \in X \).

2.2. Prove that the system of inequalities in (2.1) is equivalent to the system of three distinct triangle inequalities on three points.

2.3. (The quadrilateral inequality.) Prove the following generalization of the first inequality in (2.1). For any four points \( x, y, u, \) and \( w \) in a metric space,

\[ |d(x, y) - d(u, v)| \leq d(x, u) + d(y, v). \]

2.4. Prove inequality (2.2).

2.5. Let \( d \) be a metric on a set \( X \). Show that the following functions are also metrics on \( X \):

a) \( \bar{d} = \frac{d}{1 + d} \)

b) \( \bar{d} = \ln(1 + d) \)

c) \( \bar{d} = d^\alpha, \quad 0 < \alpha < 1 \).

2.6. For a pseudometric space \((X, d)\), we define a binary relation \( \sim \) on \( X \) by

\[ x \sim y \quad \text{if and only if} \quad d(x, y) = 0, \quad \text{for} \ x, y \in X. \]

Show that \( \sim \) is an equivalence relation on \( X \) and the quotient set \( X/\sim \) (cf. Section 1.5) is a metric space with the distance function \( \bar{d} \) given by 

\[ \bar{d}([x], [y]) = d(x, y). \]

2.7. Show that a seminorm \( p \) on a vector space \( X \) has the following properties:

a) \( p(0) = 0 \)

b) \( p(-x) = p(x) \)

c) \( p(x) \geq 0 \),

for all \( x \in X \). Also show that the function \( d(x, y) = p(x - y) \) is a pseudometric on \( X \).

2.8. Let \( d_1, \ldots, d_n \) be (pseudo)metrics on a set \( X \) and \( \lambda_1, \ldots, \lambda_n \) be nonnegative real numbers such that at least one of them is positive. Show that

\[ d = \lambda_1 d_1 + \cdots + \lambda_n d_n \]

is a (pseudo)metric on \( X \).

2.9. Let \((d_1, \ldots, d_n, \ldots)\) be a sequence of (pseudo)metrics on a set \( X \). Show that

\[ \bar{d} = \sum_k \frac{1}{2^k} \frac{d_k}{1 + d_k} \]

is a (pseudo)metric on \( X \).
2.10. Let $y$ be a point in the ball $B(x, r)$. Show that there is $r' > 0$ such that $B(y, r') \subseteq B(x, r)$. Conclude that an open ball is an open set.

2.11. Give an example of a metric space $(X, d)$ and open balls $B(x_1, r_1)$ and $B(x_2, r_2)$ in $X$ such that $B(x_2, r_2)$ is a proper subset of $B(x_1, r_1)$ and $r_2 > r_1$.

2.12. Suppose that $B(x, r_1)$ is a proper subset of $B(x, r_2)$ in a metric space $(X, d)$. Show that $r_1 < 2r_2$.

2.13. For points $x_1$ and $x_2$ in a metric space and positive numbers $r_1$ and $r_2$, prove that

a) $d(x_1, x_2) \geq r_1 + r_2$ implies $B(x_1, r_1) \cap B(x_2, r_2) = \emptyset$.

b) if $d(x_1, x_2) \leq r_1 - r_2$, then $B(x_2, r_2) \subseteq B(x_1, r_1)$.

Show that the converses of a) and b) do not necessarily hold.

2.14. Suppose that the intersection of two open balls in a metric space is not empty. Show that this intersection contains an open ball.

2.15. Show that the empty set $\emptyset$ is an open subset of every metric space.

2.16. Prove that $\text{int} \mathbb{Q} = \emptyset$.

2.17. Show that a metric space consisting of more than one point has at least two open subsets different from $\emptyset$ and $X$.

2.18. Prove that in any metric space the closure of an open ball is a subset of the closed ball with the same center and radius:

$$
\overline{B(x, r)} \subseteq \overline{B}(x, r).
$$

2.19. Prove Theorem 2.4.

2.20. Let $A$ and $B$ be sets in a metric space. Show that

a) $\overline{A} \subseteq \overline{B}$ if $A \subseteq B$.

b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

c) $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$.

Give an example of a proper inclusion in c).

2.21. Let $X$ be a subspace of a metric space $Y$ and $Z$ a subset of $X$. Show that $Z$ is closed in $X$ if and only if there is a closed subset $F$ of $Y$ such that $Z = F \cap X$.

2.22. Show that an open set in a metric space $(X, d)$ is also open in the space $(X, \tilde{d})$, where $\tilde{d} = d/(1 + d)$ (cf. Exercise 2.5a). Is the converse true?

2.23. Show that the metric space $\mathbb{C}$ is separable.
2.24. Prove that a discrete metric space is separable if and only if it is countable.

2.25. Show that \( x \) is an accumulation point of a subset \( A \) of a metric space if and only if \( x \in A \setminus \{ x \} \).

2.26. Let \( (x_n) \) be a sequence of points in a metric space \( X \) such that the set \( \{ x_n : n \in \mathbb{N} \} \) is finite. Show that \( (x_n) \) contains a convergent subsequence.

2.27. Show that a point \( x \) in a metric space \( X \) is the limit of a subsequence of a sequence \( (x_n) \) if and only if every neighborhood of \( x \) contains infinitely many terms of \( (x_n) \).

2.28. Let \( (x_n) \) be a convergent sequence in a metric space \( X \) and \( x = \lim x_n \). Show that every subsequence \( (x_{n_k}) \) of \( (x_n) \) converges to the same limit \( x \).

2.29. If \( (x_n) \) is Cauchy and has a convergent subsequence \( (x_{n_k}) \) with limit \( x \), show that \( (x_n) \) converges to the same limit \( x \).

2.30. Show that a Cauchy sequence is bounded.

2.31. Suppose that \( (x_n) \) is a Cauchy sequence in a metric space \( (X, d) \). Show that there is a subsequence \( (x_{n_k}) \) of \( (x_n) \) such that
\[
d(x_{n_k}, x_{n_m}) < 1/2^k, \quad \text{for all } m > k.
\]

2.32. Show that a metric space is a singleton if and only if every bounded sequence is convergent.

2.33. Show that a sequence \( (x_n) \) in a discrete metric space \( X \) converges to \( x \in X \) if and only if there is \( N \in \mathbb{N} \) such that \( x_n = x \) for all \( n > N \).

2.34. a) Show that the sequence \( (1/n) \) diverges in the metric space \( (0,1) \) with the distance function inherited from \( \mathbb{R} \).
   
   b) Show that this sequence is a Cauchy sequence in \( (0,1) \).

2.35. Let \( (c_n) \) be a sequence of positive rational numbers such that \( c_{k+1} < c_k \) for all \( k \in \mathbb{N} \). Show that \( \sum_{k=1}^{\infty} (-1)^{k-1} c_k = r \) implies
\[
0 < \left| r - \sum_{k=1}^{n} (-1)^{k-1} c_k \right| < c_{n+1}, \quad \text{for all } n \in \mathbb{N}.
\]

2.36. Let \( X \) be a finite set and \( (X, d) \) be a metric space. Show that this space is complete.

2.37. Let \( B(x, r) \) be an open ball in a metric space \( (X, d) \). Show that \( \overline{B(x, r)} \) is a bounded set of the diameter not greater than \( 2r \).
2.38. Show that the condition diam($A_n$) → 0 in Theorem 2.9 is essential. (Give an example of a complete metric space and a nested family of closed sets in it with an empty intersection.)

2.39. Prove the converse of Theorem 2.9 (Royden and Fitzpatrick (2010), Section 9.4).

2.40. Let $a$ be a point in a metric space $(X,d)$. Show that the function $f(x) = d(x,a)$ is uniformly continuous on $X$, that is, for a given $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon,$$

for all $x, y \in X$ such that $d(x,y) < \delta$.

2.41. For a nonempty subset $A$ of a metric space $(X,d)$ and a point $x \in X$, let

$$\text{dist}(x, A) = \inf \{d(x, a) : a \in A\}.$$

Show that the function $f(x) = \text{dist}(x, A)$ is continuous.

2.42. Prove that a subset $U$ of a metric space $X$ is open if and only if there is a continuous function $f : X \to \mathbb{R}$ such that $U = \{x \in X : f(x) > 0\}$.

2.43. Describe the completion of a discrete metric space.

2.44. If $X$ and $Y$ are isometric metric space and $X$ is complete, show that $Y$ is complete.

2.45. Let $(X,d)$ be a metric space and let $\tilde{d} = d/(1 + d)$ (cf. Exercise 2.5a). Show that $(X,d)$ is complete if and only if $(X,\tilde{d})$ is complete.

2.46. Define $d(x,y) = |\tan^{-1} x - \tan^{-1} y|$ on $\mathbb{R}$. Show that $(\mathbb{R}, d)$ is an incomplete metric space and find its completion.

2.47. Let $X$ be the set of nonempty open intervals in $\mathbb{R}$ and $d$ be defined by

$$d((x, y), (u, v)) = (y - x) + (v - u).$$

(Here, $(x, y)$ and $(u, v)$ are open intervals in $\mathbb{R}$.) Show that $(X,d)$ is incomplete metric space and find its completion.

2.48. Give an example of an incomplete metric space $X$ and a sequence $(X_n)$ of open dense subsets of $X$ with the empty intersection.

2.49. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be lower semi-continuous on $\mathbb{R}$ if the set $\{x : f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$. Let $f$ be a lower semi-continuous function on $\mathbb{R}$. Show that for every open set $U \subseteq \mathbb{R}$ there is an open subset $V \subseteq U$ on which $f$ is bounded. (Hint: use the Baire Category Theorem.)
2.50. Let \((X, d)\) be a pseudometric space such that \(d(x, y) = 0\) for some pair of distinct points \(x, y\) in \(X\). Show that there are no disjoint neighborhoods of \(x\) and \(y\) in the topological space \(X\).

2.51. Let \((X, \mathcal{T})\) be a topological space and \(X'\) be a nonempty subset of \(X\). Show that

\[ \mathcal{T}' = \{ U \cap X' : U \in \mathcal{T} \} \]

is a topology on \(X'\).

2.52. Prove Theorem 2.22.

2.53. Give an example of two homeomorphic metric spaces that are not isometric.

2.54. Let \((X_1, \mathcal{T}_1)\) and \((X_2, \mathcal{T}_2)\) be topological spaces. Show that the set

\[ \{ U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2 \} \]

is a base of topology on \(X_1 \times X_2\). (Hint: use Theorem 2.22.)