COMPUTATIONS IN THE MODULI SPACE OF CUSPIDAL CURVES

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Mathematics

by

Joel Gallegos

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CERTIFICATION OF APPROVAL

I certify that I have read *COMPUTATIONS IN THE MODULI SPACE OF CUSPIDAL CURVES* by Joel Gallegos and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Dustin Ross  
Assistant Professor of Mathematics

Emily Clader  
Assistant Professor of Mathematics

Federico Ardila  
Professor of Mathematics
Algebraic curves are fundamental objects in algebraic geometry, and the moduli space of curves tell us how algebraic curves vary in families. The moduli space of smooth curves, denoted $M_{g,n}$, parameterizes families of smooth curves of genus $g$ with $n$ marked points, but this space is not compact. There is a well-known compactification of the moduli space, denoted $\overline{M}_{g,n}$, that allows for stable nodal curves. This compactification is a choice, which means it not unique to $M_{g,n}$. Another compactification of $M_{g,n}$ allows for cuspidal curves as well; we denote this compactification by $\tilde{M}_{g,n}$. A theorem of Faber and Pandheripande provides a closed formula for integrals of the cotangent line classes against the Chern classes of the Hodge bundle on $\overline{M}_{g,1}$. The purpose of this project is to investigate if there is an analogous situation that happens on $\tilde{M}_{g,1}$. We explore an algorithm for computing cuspidal Hodge integrals, and we find surprisingly that the Faber-Pandharipande theorem remains unchanged in the cuspidal case, which we prove in special cases.

I certify that the Abstract is a correct representation of the content of this thesis.
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Chapter 1

Introduction

In this section, we introduce the moduli space of curves, introduce the Faber-Pandharipande Theorem, and state our main results.

1.1 Curves

**Definition 1.1.** A *curve* is a compact, complex algebraic variety of dimension one.

Any smooth curve over the complex numbers is topologically a smooth oriented surface over the real numbers. Therefore, by classification of surfaces, it is a $g$-holed torus for some $g$, and we define the *genus* of this curve to be $g$.

**Example 1.1.** An example of a curve is $V(zy^2 - x^3 + xz^2) \subset \mathbb{P}^2 \subset \mathbb{C}$. This is the solution set to $\{0 = zy^2 - x^3 + xz^2\}$. This is topologically equivalent to a genus 1 torus in $\mathbb{P}^2$. If we set $z = 1$, then the affine points are $V(y^2 - x^3 + x^2) \subset \mathbb{A}^2 \subset \mathbb{C}$, and we still have the genus 1 torus minus a point (a point at infinity). By restricting to only
real solutions, we are considering a slice of the variety in $A^2_R$.

![Figure 1.1: Real Affine Solutions](image1)

![Figure 1.2: Projective Complex Solutions](image2)

In Figure 1.1 we show the graph of the real affine solutions of our variety when we let $z = 1$ and Figure 1.2 shows us what our variety looks like in complex projective space.

**Definition 1.2.** A *family of curves* is a flat morphism of algebraic varieties

$$
\begin{array}{c}
X \\
\downarrow \pi \\
B
\end{array}
$$

such that for any $b \in B$, $\pi^{-1}(b) \subset X$ is a curve.

**Example 1.2.** Consider the curve from before, but now with an extra parameter and we have $V(zy^2 - x^3 + txz^2) \subset \mathbb{P}_{x,y,z}^2 \times \mathbb{C}_t$. It follows that we have a family of curves as $t$ varies,

$$
\begin{array}{c}
V(zy^2 - x^3 + txz^2) \\
\downarrow \pi \\
\mathbb{C}_t
\end{array}
$$

where for each $t \in \mathbb{C}_t$ we have that $\pi^{-1}(t)$ is a genus 1 curve.
**Definition 1.3.** A *marked curve* with $n$ points is a curve with $n$ distinct, labeled points on the curve.

**Definition 1.4.** A family of marked curves with $n$ marked points is a flat morphism of algebraic varieties with $n$ disjoint sections

\[ X \xrightarrow{\pi} \sigma \]

such that for any $b \in B$, $\pi^{-1}(b) \subset X$ is a curve with $\sigma(b) = (\sigma_1(b), ..., \sigma_n(b))$ as the marked points on that curve.

The following result tells us that families of marked curves are parameterized by a nice geometric space.

**Theorem 1.1** (Deligne–Mumford [1]). There exists a smooth, Deligne–Mumford stack of dimension $3g - 3 + n$ that parameterizes families of smooth genus $g$, $n$ marked curves up to isomorphism. This is the moduli space of curves, denoted $M_{g,n}$.

**Remark.** Deligne-Mumford stacks are something we will not carefully define, refer to [1]. For our purposes, $M_{g,n}$ can be thought of as a smooth variety.

For every family of smooth marked curves, there is a unique map

\[ \phi : B \to M_{g,n} \]
such that $\phi(b)$ parameterizes the marked curve $(C, p_1, \cdots, p_n)$ where $C = \pi^{-1}(b)$ and $\sigma_i(b) = p_i$. Thus, $M_{g,n}$ is made up of all these family of curves, where each point in $M_{g,n}$ represents some curve in one of the families. When we talk about a point in $M_{g,n}$ parameterizing a marked curve, this point is parameterizing the isomorphism class of that marked curve [7].

1.1.1 Compactifying the Moduli Space

We begin with an example that will highlight the reason $M_{g,n}$ is not compact.

**Example 1.3.** Let $X = V(y^2z - x^3 + 3xz^2 - tz^3)$ and $B = \mathbb{C} \setminus \{4\}$ where $t \in \mathbb{C}$. Then $X$ is the solution set to $\{0 = y^2z - x^3 + 3xz^2 - tz^3\}$, and we have the family

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & B \\
\downarrow & & \\
B & & 
\end{array}
$$

with the unique map $\phi : B \to M_{g,n}$. Looking at affine points by setting $z = 1$, as $t \to 4$ we have $V(y^2 - x^3 + 3x^2 - 4)$,
and we have a singularity point at \((2, 0)\). This curve is no longer smooth, but it is a “limit point” for the family of curves. This will need to be a point which we want to include in the compactification of \(M_{g,n}\).

One way to compactify \(M_{g,n}\) is to allow stable nodal curves, stable curves are complex algebraic curves with only one type of singularity, nodes. This compactification is called the Deligne–Mumford compactification of \(M_{g,n}\), denoted \(\overline{M}_{g,n}\).

**Definition 1.5 ([12]).** A stable curve \(C\) with genus \(g\) and \(n\) marked points is a compact complex algebraic curve such that

- the only singularities on \(C\) are simple nodes (locally looks like \(xy = 0\));
- the marked points are distinct and do not coincide with the nodes;
- every genus 0 irreducible component of the curve has at least 3 special points.
- every genus 1 irreducible component of the curve has at least 1 special point.

**Remark.** By special points we mean either a marked point or a node.

**Theorem 1.2** (Deligne–Mumford [6]). There is a compact moduli space of curves, $\overline{M}_{g,n}$, that allows for stable nodal curves, and $\dim \overline{M}_{g,n} = 3g - 3 + n$.

**Definition 1.6.** The boundary of $\overline{M}_{g,n}$ is defined to be the elements in $\overline{M}_{g,n} \setminus M_{g,n}$.

**Example 1.4.**

- Consider $\overline{M}_{2,1}$, a typical point is a curve $C$ with genus 2 and 1 marked point. If we were to lasso around the curve between the 2 genera and squeeze, then we would be adding a node to our curve $C$. This would be a “limit point” of the family of marked curves with different variations of how hard we squeeze. Our curve $C$ now has two irreducible parts attached at a node making $C$ also an element of $\overline{M}_{1,1} \times \overline{M}_{1,2}$. This is an example of an element in the boundary of $\overline{M}_{2,1}$. The dimension of $\overline{M}_{2,1}$ is $3(2) - 3 + (1) = 4$.

We shift our attention to an alternate compactification of $M_{g,n}$ allowing for pseudo-stable curves giving us a pseudo-stable compactification.

**Definition 1.7** ([9]). A pseudo-stable curve $C$ with genus $g$ and $n$ marked points is a compact complex algebraic curve such that

- it only has nodes and ordinary cusps as singularities (locally looks like $xy = 0$ or $y^2 = x^3$);
• the marked points are distinct and do not coincide with the nodes;
• every irreducible genus 0 component has at least 3 special points;
• every irreducible genus 1 component has at least 2 special points.

**Theorem 1.3** (Schubert [9]). There is a compact moduli space of curves, \( \tilde{M}_{g,n} \), that allows for pseudo-stable curves, and \( \dim \tilde{M}_{g,n} = 3g - 3 + n \).

There is a map

\[ f : \overline{M}_{g,n} \to \tilde{M}_{g,n} \]

which collapses irreducible genus 1 components with only one special point into a cusp. If the curve \( C \in \overline{M}_{g,n} \) has no irreducible genus 1 component with only one special point, then it is mapped trivially to itself because it is still pseudo-stable. If \( C \) does have an irreducible genus 1 component that has only one special point, then it is not pseudo-stable and we will need to stabilize it.

Denote \( I \) as the irreducible genus 1 component with only one special point (which means the special point is a node) and \( C' \) be the rest of the curve \( C \) that is attached to \( I \) at a node. To stabilize \( C \) (pseudo-stabilize), then we take \( I \) and collapse the whole irreducible genus 1 component to a cusp where locally we have \( y^2 = x^3 \). After, repeat the process with \( C' \) to check for any other irreducible genus 1 components with no marked points. This is the stabilization process of the curves going from \( \overline{M}_{g,n} \) to \( \tilde{M}_{g,n} \). The following is an example of a cuspidal curve.
Comparing back to Figure 1.3, around the node it locally looks like $xy = 0$ where as in Figure 1.4 now looks like $y^2 = x^3$ locally. The interesting question that we want to ask now is what are the differences or similarities in the Chow rings of the two compactified moduli spaces. This is quite a difficult question to ask, which pushes us to try to intersect classes of the Chow ring with other specific classes to examine that instead. Comparing intersection numbers of $\overline{M}_{g,n}$ and $\tilde{M}_{g,n}$ could give us information about some classes in both Chow rings that highlight differences or similarities. It has been shown by Faber-Pandharipande over $\overline{M}_{g,1}$ there is a closed formula for a certain class of intersection numbers. We want to take that same theorem from Faber-Pandharipande and study it over $\tilde{M}_{g,1}$ to look at its intersection numbers for comparison in hopes of finding a closed formula as well.
1.1.2 Computing Intersection Numbers

With each vector bundle over a variety, one can define specific classes, Chern classes, in its Chow ring. In the next section, we carefully define λ-classes with respect to the Hodge bundle and ψ-classes with respect to the line bundle \( L \). We intersect these two specific classes to look at their intersection numbers in both compactifications. By a theorem of Faber-Pandharipande, there is a closed formula that describes the intersection numbers on \( \overline{M}_{g,1} \).

**Theorem 1.4** (Faber-Pandharipande [3]).

\[
1 + \sum_{g \geq 0} \sum_{i=1}^{g} \left( \int_{\overline{M}_{g,1}} \psi^{2g-2+i} \lambda_{g-i} \right) t^{2g} k^{i} = \left( \frac{t/2}{\sin(t/2)} \right)^{k+1}.
\]

Without defining anything precisely, the left side is a generating function holding the intersection numbers of different values for \( g \) and \( i \). This has a closed formula, on the right, which we will examine more closely with an example.

**Example 1.5.** For this example we will let \( k = 0 \), which implies \( i = 0 \) giving us

\[
1 + \sum_{g \geq 0} \left( \int_{\overline{M}_{g,1}} \psi^{2g-2} \lambda_{g} \right) t^{2g}.
\]

From here, we use Macaulay2 [4] to compute some values using the HodgeIntegrals package [11].
• For $g = 1$,
\[
\int_{\overline{M}_{1,1}} \psi^0 \lambda_1 = \int_{\overline{M}_{1,1}} \lambda_1 = \frac{1}{24}.
\]

• For $g = 2$,
\[
\int_{\overline{M}_{2,1}} \psi^2 \lambda_2 = \int_{\overline{M}_{2,1}} \psi^2 \lambda_2 = \frac{7}{5760}.
\]

Remembering $k = 0$, we Taylor expand the right hand side,
\[
h(0) = \frac{t/2}{\sin(t/2)} = 1 + \frac{t^2}{24} + \frac{7t^4}{5760} + \frac{31t^6}{967680} + \cdots.
\]

We see that the coefficients are exactly the values we got using Macaulay2. Using different values for $g$ and $i$, we get Table 1.1 where the coefficients of the different derivatives of $h(k)$ evaluated at 0 are the same values as well.

<table>
<thead>
<tr>
<th>$i/g$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1/24</td>
<td>7/5760</td>
<td>31/967680</td>
<td>127/154828800</td>
</tr>
<tr>
<td>1</td>
<td>1/24</td>
<td>1/480</td>
<td>41/580608</td>
<td>13/6220800</td>
</tr>
<tr>
<td>2</td>
<td>1/1152</td>
<td>7/138240</td>
<td>1357/696729600</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1/82944</td>
<td>1/1244160</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>1/79650624</td>
<td></td>
</tr>
</tbody>
</table>

We made a similar table for the computed intersection values over $\overline{M}_{g,1}$. To our surprise, the tables of the intersection numbers over both compactifications were identical. The values in Table 1.1 are also for the computed intersection values over $\overline{M}_{g,1}$, up to genus 6, which leads us to our conjecture.
Conjecture 1.1. For all $i$ and $g > 1$,

$$\int_{\tilde{M}_{g,1}} \tilde{\lambda}_{g-i} \tilde{\psi}_1^{2g-2+i} = \int_{\tilde{M}_{g,1}} \lambda_{g-i} \psi_1^{2g-2+i}$$

For some cases, specifically when $g \leq i + 3$, we have a proven theorem which gives us strong evidence, along with our tables, that our conjecture could be true.

Theorem 1.5. For $g > 1$,

$$\int_{\tilde{M}_{g,1}} \tilde{\lambda}_{g-i} \tilde{\psi}_1^{2g-2+i} = \int_{\tilde{M}_1} \lambda_{g-i} \psi_1^{2g-2+i}$$

for $g \leq i + 3$. 

Chapter 2

Pre-requisites

Some terms we neglected to define in the introduction, along with some key topics, will now be explained in further detail.

2.1 Chow Ring

For this section, $X$ will be a smooth, quasi projective variety of dimension $n$. There will be certain terms or ideas that we will go in depth and others that we will informally discuss. In either case, refer to [2] for further details.

**Definition 2.1.** The *group of cycles*, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties of $X$. This is graded by dimension,

$$Z(X) = \bigoplus_{k \geq 0} Z_k(X)$$

where $Z_k(X)$ is the group of cycles that are formal linear combinations of subvarieties
of dimension \( k \), also called \( k \)-cycles.

A cycle is a sum of irreducible subvarieties of \( X \). Hence, a cycle \( Z \) can be written as

\[
Z = \sum a_i Y_i
\]

where \( Y_i \subset X \) are irreducible subvarieties and \( a_i \) are rational coefficients for each \( i \). It should be noted that we will be interchanging subvarieties with cycles. A subvariety will always correspond to a cycle, but a cycle will not always be a subvariety because the coefficients can be anything including negative values. A subvariety can be thought of as a union of its irreducible subvarieties, analogously as a cycle made up of a linear combination of cycles of its irreducible subvarieties.

**Definition 2.2** ([2]). Consider \( A, B \in Z_k(X) \). We say \( A \) and \( B \) are rationally equivalent \((A \sim_{\text{rat}} B)\) if there is some \( C \in Z_{k+1}(X \times \mathbb{P}^1) \) such that

\[
C \cap (X \times \{0\}) = A
\]

and

\[
C \cap (X \times \{\infty\}) = B.
\]

**Definition 2.3.** The Chow group of \( X \) is the group of cycles of \( X \) modulo rational equivalence. In other words, it is the group of rationally equivalent classes of cycles
of $X$ denoted
\[ A(X) = \frac{Z(X)}{\sim_{\text{rat}}} . \]

We eventually want to have a ring structure on the Chow group where the product is defined as the intersection of classes. But we want the intersection to be well defined. For intuition consider $\mathbb{R}^3$ and two subvarieties, [plane] and [line]. If we intersect both classes, then we expect and want
\[ \text{[plane]} \cdot \text{[line]} = \text{[plane } \cap \text{ line]} = \text{[point]} . \]

Here these two subvarieties intersect *transversely*, and the next issue that we will address soon is when they do not intersect transversely.

**Definition 2.4 ([2]).** Consider the subvarieties $A, B \subset X$. We say $A$ and $B$ intersect transversely at point $p$ if $A, B$ and $X$ are all smooth at $p$ and the tangent spaces to $A$ and $B$ at $p$ together span the tangent space of $X$. In other words,
\[ T_pA \oplus T_pB = T_pX. \]

We say that the subvarieties $A, B \subset X$ are *generically transverse* to each other if they meet transversely at a general point in each component $C$ of $A \cap B$. We say cycles $Y = \sum a_iA_i, Z = \sum b_jB_j \subset Z(X)$ are generically transverse if $A_i$ is generically transverse to $B_j$ for each $j$. 
We come back to the issue from earlier, suppose $A, B \in A(X)$ do not intersect transversely. Then it could be

$$[\text{plane}] \cap [\text{line}] = [\text{line}].$$

where actually the line is in the plane meaning the two classes do not intersect transversely. The next result allows us to fix such a situation.

**Lemma 2.1 ([2]).** Let $X$ be a smooth, quasi-projective variety.

- For every $a, b \in A(X)$ there are generically transverse cycles $A, B \in Z(X)$ such that $[A] = a$ and $[B] = b$.

- The class $[A \cap B]$ is independent of the choice of such cycles $A$ and $B$.

This lemma is key in defining the Chow ring because it says we can always intersect classes of varieties by choosing the right representative and the intersection is independent of that choice.

**Definition 2.5 ([2]).** Let $X$ be a smooth quasi-projective variety. There is a unique product structure on $A(X)$ such that if we have $A, B \in A(X)$ that are generically transverse, then

$$[A] \cdot [B] = [A \cap B].$$
This structure makes the Chow ring of $X$, denoted $A^*(X)$, where

$$A(X) = A^0(X) \oplus A^1(X) \oplus \cdots \oplus A^n(X),$$

an associative, commutative ring, graded by co-dimension.

**Remark.** For each $i$, $A^i(X)$ consists of all the co-dimension $i$ cycles of $X$ under rational equivalence.

With each $\gamma \in A^*(X)$, where $X$ is some variety of dimension $n$, we can write it as

$$\gamma = \gamma^0 \oplus \gamma^1 \oplus \gamma^2 \oplus \cdots \oplus \gamma^n$$

Looking at the co-dimension $n$ part, we want to be able count the points it has that lie within it, $\gamma^n = \sum a_ip_i$ where $p_i \in X$ are classes of points. In other words, $\gamma^n$ is a linear combination of points and we define the integration map to count the coefficients.

**Definition 2.6** (Integration Map). Consider $\gamma \in A^*(X)$ of some variety $X$ of dimension $n$ where

$$\gamma = \gamma^0 \oplus \gamma^1 \oplus \gamma^2 \oplus \cdots \oplus \gamma^n$$

and $\gamma^n = \sum a_ip_i$. Then

$$\int_X \gamma = \sum a_i.$$

**Example 2.1** (Bezout’s Theorem).
Consider two curves $C_d, C_e \in \mathbb{P}^2_C$ of degree $d$ and $e$, respectively. Then we have two hypersurfaces in $\mathbb{P}^2_C$, defined by homogeneous polynomials of degree $d$ and $e$, intersect at $d \cdot e$ points with multiplicity counted. Thus, the integral is defined to count the same points,

$$\int_{\mathbb{P}^2_C} [C_d][C_e] = d \cdot e.$$

### 2.2 Vector Bundles

In order to talk about $\lambda$ and $\psi$ classes we must first define what a vector bundle is.

**Definition 2.7 ([10]).** A vector bundle $E$ of rank $n$ on an algebraic variety $X$ is a morphism 

$$\pi : E \rightarrow X$$

where $E$ is an algebraic variety such that there is an open cover of $X = \bigcup_{\alpha} U_{\alpha}$ where $\pi^{-1}(U_{\alpha}) = U_{\alpha} \times \mathbb{C}^n$ and 

$$\pi|_{\pi^{-1}(U_{\alpha})} : U_{\alpha} \times \mathbb{C}^n \rightarrow U_{\alpha}$$

is the projection onto the first factor.

**Remark.** This is saying that the vector bundle is locally trivial, or has a local trivialization.

With each vector bundle we can define maps going back, these are called sections
of the vector bundle.

**Definition 2.8.** Let \((E, X, \pi)\) be a vector bundle and \(U \subset X\). A section of a vector bundle over \(U\) is a morphism

\[ s : U \to E \]

such that \(\pi \circ s\) is the identity map on \(U\).

*Remark.* Every vector bundle admits at least one section, the zero section.

**Example 2.2.** Let \(X \subset \mathbb{P}^n\) and let \(E = X \times \mathbb{C}\), then the trivial line bundle on \(X\) will be

\[ \pi : E \to X \]

\[ (x, a) \mapsto x. \]

The sections of the trivial line bundle are all the morphisms \(s : x \mapsto (x, f(x))\) where \(f : U \to \mathbb{C}\) is a regular function on \(U \subset X\).

The group of sections is a vector space because we can add any two sections of a vector bundle and closed under scalar multiplication [10]. We now turn our focus to some specific vector bundles. There will be two important vector bundles for us, the Hodge bundle and the line bundle \(L_i\), over \(\overline{M}_{g,n}\).

**Definition 2.9** ([5]). Consider \((C, p_1, \ldots, p_n) \in \overline{M}_{g,n}\), the cotangent line to \(C\) at point \(p_i\) is a one dimensional vector space; these spaces “patch together” to give the line bundle \(\mathbb{L}_i\), called the *ith tautological line bundle*. 
Considering the vector bundle $L_i$ on $\overline{M}_{g,n}$, we can visually describe the fiber above each curve in our space:

\[
\begin{array}{c}
T_C^\vee|_{p_i} \hookrightarrow L_i \\
\downarrow \downarrow \\
(C, p_1, \ldots, p_n) \hookrightarrow \overline{M}_{g,n}
\end{array}
\]

Above each point in our base, $\overline{M}_{g,n}$, the fiber of that point is the one-dimensional vector space corresponding to the cotangent line to $C$ and the point $p_i$.

**Example 2.3.**

- Consider $\overline{M}_{0,4}$ and $L_3$, then the fiber over some point $(\mathbb{P}^1, 0, 1, p_3, \infty) \in \overline{M}_{0,4}$ is the co-tangent line of $\mathbb{P}^1$ at $p_3$, denoted $T_C^\vee|_{p_3}$ [7].

Given a genus $g$ smooth curve $C$ with the cotangent line bundle $T_C^\vee$, we can look at the global sections of this vector bundle, $\Gamma(C, T_C^\vee)$. It can be shown that $\dim(\Gamma(C, T_C^\vee)) = g$, refer to [8].

**Definition 2.10** ([5]). The rank $g$ vector spaces, $\Gamma(C, T_C^\vee)$, “patch together” and extend to the boundary to give a rank $g$ vector bundle $E$, called the Hodge bundle, on $\overline{M}_{g,n}$.

**Remark.** The last two definitions also define vector bundles $\tilde{L}_i$ and $\tilde{E}$ over $\overline{M}_{g,n}$.

Again, considering a smooth $C \in \overline{M}_{g,n}$, the fiber of $C$ is defined to be the global sections of $T_C^\vee$. In other words,
As before with $\mathbb{L}_i$, we will do a few examples to have a better feel for the definition.

**Example 2.4.**

- Let $g = 0$, then $E$ is the trivial bundle over $\overline{M}_{0,n}$ because $\dim (\Gamma (C, T_C^\vee)) = 0$. This means $E$ is a vector bundle where each vector space above each curve is a rank 0 vector space.

- Let $g = 1$, then $E$ is a line bundle over $\overline{M}_{1,n}$ because $\dim (\Gamma (C, T_C^\vee)) = 1$. In fact, $E = \mathbb{L}_1$ on $\overline{M}_{1,1}$ [12].

### 2.3 Characteristic classes

Given a rank $k$ vector bundle $V$ on $X$, the Chern classes are co-dimension $i$ classes:

$$c_i(V) \in A^i(X) \text{ for } 0 \leq i \leq k.$$  

**Definition 2.11.** Consider general sections $s_1, \ldots, s_{k-i+1} \in \Gamma(X, V)$ of the vector bundle $V$ on $X$. Then the $i^{th}$ Chern class is

$$c_i(V) = \{ x \in X : \sum a_is_i(x) = 0 \text{ for some } a_i \text{ not all zero} \},$$
and the total Chern class is $c(V) = c_0(V) \oplus c_1(V) \oplus \cdots \oplus c_k(V)$.

*Remark.* General sections do not always exist, but there is a way to define Chern classes even when they do not.

**Example 2.5.**

1. Consider the trivial bundle $V$ of rank $r$ on some projective variety $X$, then $c(V) = c_0(V) = 1$ since $c_i(V) = 0$ for $i \neq 0$ because any regular function on a projective variety is a constant function. Given any section on $X$, it will be a constant function which means $i$ general linearly independent sections will never be linearly independent for every value in $X$ [10].

2. Consider a line bundle $L$ over a variety $X$, then we only have two Chern classes, $c_0(V) + c_1(V)$. Thus,

$$c_0(V) = [X] = 1$$
$$c_1(V) = \text{[zeroes of a regular section]}$$

and the total Chern class is $c(V) = c_0(V) + c_1(V) \in A^*(X)$.

**Definition 2.12.** Given $L_i$ over $\overline{M}_{g,n}$, then

$$\psi_i = c_1(L_i) \in A^1(\overline{M}_{g,n})$$

for $1 \leq i \leq n$. 
**Definition 2.13.** Given $E$ of rank $g$ over $\mathcal{M}_{g,n}$, then

$$\lambda_j = c_j(E) \in A^j(\mathcal{M}_{g,n})$$

for $1 \leq j \leq g$.

We want to be able to compare Chern classes, but this is not easy. This leads us to define Chern characters because this, as we will see, is easier to compare.

**Definition 2.14.** Consider the variety $X$ and the vector bundle $V$ of rank $r$. Then

$$c(V) = 1 + c_1(V) + \cdots + c_r(V)$$

$$= \prod_{i=1}^{r}(1 + \rho_i)$$

$$= 1 + \sum_{i=1}^{r} \rho_i + \sum_{i=0}^{r} \sum_{i<j} \rho_i \rho_j + \cdots + (\rho_1 \rho_2 \cdots \rho_r)$$

where we call $\rho_i$ a Chern root [12].

**Example 2.6.** Let $X$ be a variety with a vector bundle $V$.

- $\dim(V) = 1$: 

$$c(V) = c_0(V) + c_1(V) = 1 + c_1(V) = 1 + \rho_1.$$
• \( \dim(V) = 2 \):

\[
c(V) = 1 + c_1(V) + c_2(V) = \Pi_{i=1}^{2}(1 + \rho_i) = 1 + (\rho_1 + \rho_2) + \rho_1\rho_2.
\]

where \( c_1(V) = \rho_1 + \rho_2 \) and \( c_2(V) = \rho_1\rho_2 \).

**Definition 2.15 ([12]).** Consider the variety \( X \) with the vector bundle \( V \) of rank \( r \) over it. The Chern character of \( V \) is

\[
ch(V) = \sum_{i=1}^{r} e^{\rho_i}.
\]

**Remark.** Notice that the highest Chern class is \( c_r(V) \), but there is no highest Chern character.

**Example 2.7.** Let \( X \) be a variety with the vector bundle \( V \).

• \( \dim(V) = 1 \): Using Definition 2.15,

\[
ch(V) = e^{\rho_1} = 1 + \rho_1 + \frac{(\rho_1)^2}{2!} + \frac{(\rho_1)^3}{3!} + \cdots = 1 + c_1(V) + c_2(V) + c_3(V) + \cdots.
\]

Using the previous example, it follows with some back substitution that the
first four Chern characters are

\[ ch_1(V) = c_1(V) \]
\[ ch_2(V) = \frac{(c_1(V))^2}{2!} \]
\[ ch_3(V) = \frac{(c_1(V))^3}{3!} \]
\[ ch_4(V) = \frac{(c_1(V))^4}{4!} . \]

\[ \bullet \ \text{dim}(V) = 2 : \text{Using Definition 2.15}, \]
\[ ch(V) = \sum_{i=1}^{2} e^{\rho_i} \]
\[ = 2 + (\rho_1 + \rho_2) + \frac{(\rho_1)^2 + (\rho_2)^2}{2!} + \frac{(\rho_1)^3 + (\rho_2)^3}{3!} + \cdots \]
\[ = 2 + ch_1(V) + ch_2(V) + ch_3(V) + \cdots . \]

Using the previous example, it follows with some back substitution that the
first four Chern characters are

\[
\begin{align*}
ch_1(V) &= c_1(V) \\
ch_2(V) &= \frac{(c_1(V))^2 - 2c_2(V)}{2!} \\
ch_3(V) &= \frac{(c_1(V))^3 - 3c_1(V)c_2(V)}{3!} \\
ch_4(V) &= \frac{(c_1(V))^4 - 4c_1(V)^2c_2(V) + 2c_2(V)^2}{4!}.
\end{align*}
\]

**Remark.** Notice that \( ch_0(V) = r \) whenever \( \text{dim}(V) = r \).

As the dimension of \( V \) goes up so will the difficulty of finding \( ch_i(V) \) in terms of Chern classes which increases the chances of making a mistake. Notice the total Chern class is defined in terms of Chern roots, Definition 2.14, and similarly we can do the same for Chern characters using Definition 2.15. These are examples of *elementary symmetric function* and *power sum symmetric function*, respectively.

Since \( c_k = e_k(\rho_1, \cdots, \rho_r) \) and \( ch_k = \frac{p_k(\rho_1, \cdots, \rho_r)}{k!} \), then Newton’s identity tells us

\[
c_k(V) = \frac{1}{k!} \det \begin{pmatrix}
1!ch_1(V) & 1 & 0 & 0 & \cdots & 0 \\
2!ch_2(V) & 1!ch_1(V) & 2 & 0 & \cdots & 0 \\
3!ch_3(V) & 2!ch_2(V) & 1!ch_1(V) & 3 & \cdots & 0 \\
4!ch_4(V) & 3!ch_3(V) & 2!ch_2(V) & 1!ch_1(V) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
(k!ch_k(V) & (k-1)!ch_{k-1}(V) & (k-2)!ch_{k-2}(V) & (k-3)!ch_{k-3}(V) & \cdots & 1!ch_1(V) \\
\end{pmatrix}.
\]
The first four Chern classes of a vector bundle $V$ are

\begin{align*}
  c_1 &= ch_1 \\
  c_2 &= \frac{1}{2} ch_1^2 - ch_2 \\
  c_3 &= 2ch_3 + \frac{1}{6} ch_1^2 - ch_1 ch_2 \\
  c_4 &= \frac{1}{24} ch_1^4 - \frac{1}{2} ch_2 ch_1^2 + 2ch_3 ch_1 + \frac{1}{2} ch_2^2 - 6ch_4.
\end{align*}

*Remark.* For simplicity, $c_i = c_i(V)$ and $ch_i = ch_i(V)$ with respect to the vector bundle $V$. 
Chapter 3

Analysis

3.1 Computing Cuspidal Hodge Integrals

Our goal is eventually to compute integrals using \( \tilde{\lambda} \) and \( \tilde{\psi} \) classes, but Macauly2 does not know how to operate with these. Hence, we will need translate \( \tilde{\lambda} \) and \( \tilde{\psi} \) classes into Chern characters from the Deligne–Mumford compactification. In order to do this, we consider the following morphisms.

\[
\overline{M}_{1,\ast} \times \overline{M}_{g-1,\bullet,1} \xrightarrow{j} \overline{M}_{g,1} \xrightarrow{f} \tilde{\overline{M}}_{g,1}. \tag{1}
\]

- the map \( j \) attaches the two curves at the marked points, \( \ast \) and \( \bullet \), from the curves to a node, hence there will only be one marked pointed on the \( g - 1 \) genus part of the curve in \( \overline{M}_{g,1} \).

- \( f \) trivially maps everything else in \( \overline{M}_{g,1} \) to itself that is not in the image of \( j \).
• $f$ takes everything in the image of $j$ and collapses the irreducible, non-pseudo-stable genus 1 pieces into a cusp at the attaching node.

Remark. The marked points $\{\ast, \bullet, 1\}$ are all arbitrary names for distinguishing the different marked points. Later, we will just use $L$ for the marked point on the irreducible genus 1 component instead of $\ast$. Similarly, we will use $R$ for the marked point on the irreducible genus $g - 1$ component instead of $\bullet$.

Everything in the image of $j$ has its irreducible genus 1 component with no marked points being collapsed. We want define this class of curves from the image of $j$.

Definition 3.1. Considering the morphism $j$ from the above diagram, the image of $j$ is a subvariety of $\overline{M}_{g,1}$. The image of $j$ is a subvariety of classes of curves of codimension one with an irreducible genus 1 component with no marked points attached at a node with a $g - 1$ component with 1 marked point. We denote this divisor, the image of $j$, as $D$. In other words, $D = im(j)$.

The classes, $\psi_L$ and $\psi_R$, refer to the $\psi$-classes on the left or right components, respectively. If we exclude $D$, $f$ is an isomorphism which means that integration over $\tilde{M}_{g,1}$ is the same as integrating over $\overline{M}_{g,1}$ [2].

Proposition 3.1. From the above diagram, since $f : \overline{M}_{g,1} \to \tilde{M}_{g,1}$ is an isomor-
phism away from $D$, intersection numbers can be computed on $\overline{M}_{g,1}$:

$$\int_{\overline{M}_{g,1}} \overline{\lambda}_{g-i} \overline{\psi}_1^{2g-2+i} = \int_{\overline{M}_{g,1}} f^* (\overline{\lambda}_{g-i}) f^* (\overline{\psi}_1)^{2g-2+i}.$$ 

Remembering diagram (1), since $f^* (\overline{\mathbb{L}}_i) = \mathbb{L}_i$ because the marked point is never on the irreducible genus 1 component that is collapsed, then $f^* (\overline{\psi}_i) = \psi_i$. Taking a closer look at diagram (1) with the Hodge bundle,

$$\begin{array}{ccc}
\mathbb{E}_1 & \downarrow & \mathbb{E}_g \\
\downarrow & & \downarrow \\
\overline{M}_{1,1} \times \overline{M}_{g-1,2} & \overset{j}{\longrightarrow} & \overline{M}_{g,1} \\
& \overset{f}{\longrightarrow} & \overline{M}_{g,1},
\end{array}$$

one major difference between the two varieties is the irreducible genus 1 components with no marked points. When we pull back $\overline{\lambda}$-classes with $f^*$ this becomes an issue and the difference in $\lambda$-classes and $\overline{\lambda}$-classes. Thus, it remains to compute $f^* (\overline{\lambda}_i)$, and see how the Chern characters pull back.

**Theorem 3.2** (Cavalieri–Ross–Wise, unpublished).

$$f^*(\overline{c}_i) = c_i - (-1)^i \left[ j_* \left( (1 - \psi_L) \left( e^{\psi_L + \psi_R} - 1 \right) \right) \right]_i,$$

where $\psi_L$ and $\psi_R$ are the $\psi$-classes at the node on the genus-1 component and the genus-$(g - 1)$ component, respectively.
Taking a closer look at the right hand side,

\[
e^{\psi_L + \psi_R - 1} \frac{\psi_L + \psi_R}{\psi_L + \psi_R} = \frac{(\psi_L + \psi_R) + (\psi_L + \psi_R)^2}{2!} + \ldots
\]

\[
= 1 + \frac{(\psi_L + \psi_R)}{2!} + \frac{(\psi_L + \psi_R)^2}{3!} + \ldots
\]

\[
= \sum_{i=0}^{\infty} \frac{(\psi_L + \psi_R)^i}{(i + 1)!}
\]

\[
= \sum_{i=0}^{\infty} \frac{(\psi_L \psi_R^{-1} + \psi_R)}{(i + 1)!}
\]

using the fact that \(\psi^2_L = 0\).

By replacing \(\left(\frac{e^{\psi_L + \psi_R - 1}}{\psi_L + \psi_R}\right)\), it follows

\[
j_* \left( (1 - \psi_L) \left( \frac{e^{\psi_L + \psi_R - 1}}{\psi_L + \psi_R} \right) \right) = j_* \left( (1 - \psi_L) \sum_{i=0}^{\infty} \frac{(\psi_L \psi_R^{i-1} + \psi_R^i)}{(i + 1)!} \right)
\]

\[
= j_* \left( \sum_{i=0}^{\infty} \frac{(\psi_L \psi_R^{i-1} + \psi_R^i)}{(i + 1)!} - \sum_{i=0}^{\infty} \frac{(\psi_L \psi_R^i)}{(i + 1)!} \right)
\]

\[
= j_* \left( \sum_{i=0}^{\infty} \frac{\psi_R^i - \psi_L \psi_R^{i-1}}{(i + 1)!} \right).
\]

This means we can rewrite the right hand side of Theorem 3.2

\[
f^* (\tilde{c}h_i) = ch_i - (-1)^i j_* \left( \frac{\psi_R^i - \psi_L \psi_R^{i-1}}{(i + 1)!} \right).
\]
With this we can compute the first four cuspidal Chern characters are:

\[
\begin{align*}
\tilde{c}h_1 &= ch_1 + J_1 \\
\tilde{c}h_2 &= ch_2 - J_2 \\
\tilde{c}h_3 &= ch_3 + J_3 \\
\tilde{c}h_4 &= ch_4 - J_4
\end{align*}
\]

where the first four correction terms are

\[
\begin{align*}
J_1 &= D \\
J_2 &= D \left( \frac{\psi_R - \psi_L}{2} \right) \\
J_3 &= D \left( \frac{\psi_R^2 - \psi_L \psi_R}{6} \right) \\
J_4 &= D \left( \frac{\psi_R^3 - \psi_L \psi_R^2}{24} \right)
\end{align*}
\]
Remembering $\tilde{\lambda}_i = \tilde{c}_i$ and using Newton’s identity, we have the first four $\tilde{\lambda}$-classes

\begin{align*}
\tilde{\lambda}_1 &= \tilde{c}h_1 \\
\tilde{\lambda}_2 &= \frac{1}{2} \tilde{c}^2 h_1 - \tilde{c}h_2 \\
\tilde{\lambda}_3 &= 2\tilde{c}h_3 + \frac{1}{6} \tilde{c}^3 - \tilde{c}h_1 \tilde{c}h_2 \\
\tilde{\lambda}_4 &= -6\tilde{c}h_4 + \frac{1}{24} \tilde{c}^4 + 2\tilde{c}h_1 \tilde{c}h_3 - \frac{1}{2} \tilde{c}^2 h_1 \tilde{c}h_2 + \frac{1}{2} \tilde{c}^2 h_2.
\end{align*}

Substituting what we have found, we finally have our $\tilde{\lambda}$-classes in terms of Chern characters with respect to the Deligne–Mumford compactification.
\( \tilde{\lambda}_1 = \tilde{\chi}_1 = \chi_1 + J_1 \)

\( \tilde{\lambda}_2 = \frac{1}{2} \tilde{\chi}_1^2 - \tilde{\chi}_2 = \frac{1}{2} (\chi_1 + J_1)^2 - \chi_2 + J_2 = \lambda_2 + \chi_1 J_1 + \frac{1}{2} J_1^2 + J_2 \)

\( \tilde{\lambda}_3 = 2 \tilde{\chi}_3 + \frac{1}{6} \tilde{\chi}_1^3 - \tilde{\chi}_1 \tilde{\chi}_2 \)

\( = 2 \chi_3 + 2 J_3 + \frac{1}{6} (\chi_1 + J_1)^3 - (\chi_1 + J_1) (\chi_2 - J_2) \)

\( = \lambda_3 + 2 J_3 + \frac{1}{2} \chi_1 J_1 + \frac{1}{2} \chi_1 J_2^2 + \frac{1}{6} J_1^3 + \chi_1 J_2 - \chi_2 J_1 + J_1 J_2 \)

\( \tilde{\lambda}_4 = -6 \tilde{\chi}_4 + \frac{1}{24} \chi_1^4 + 2 \tilde{\chi}_1 \tilde{\chi}_3 - \frac{1}{2} \tilde{\chi}_1 \tilde{\chi}_2 + \frac{1}{2} \tilde{\chi}_2 \)

\( = -6 (\chi_4 - J_4) + \frac{1}{24} (\chi_1 + J_1)^4 + 2 (\chi_1 + J_1) (\chi_3 + J_3) \)

\( - \frac{1}{2} (\chi_1 + J_1)^2 (\chi_2 - J_2) + \frac{1}{2} (\chi_2 - J_2)^2 \)

\( = \lambda_4 + 6 J_4 + 2 \chi_1 J_3 + 2 J_1 J_3 - \chi_1 \chi_2 J_1 - \frac{1}{2} \chi_2 J_1^2 + \chi_1 J_2^2 + \chi_1 J_1 J_2 + \frac{1}{2} J_1^2 J_2 \)

\( + \frac{1}{2} \chi_2^2 - \chi_2 J_2 + \frac{1}{2} J_2 + \frac{1}{6} \chi_1 J_1 + \frac{1}{12} \chi_1 J_1^2 + \frac{1}{6} \chi_1 J_1^3 + \frac{1}{6} \chi_1 J_2^2 + \frac{1}{24} J_1^4 \)

With computing integration of our \( \tilde{\lambda} \)-classes with specific \( \psi \)-classes, we must first say how terms like \( \chi_i J_k \) or \( J_i J_k \) multiply.

**Proposition 3.3.** [12]

\[
\chi_i J_k = D(\chi_{i,L} + \chi_{i,R}) \left( \frac{\psi_R^k - \psi_L \psi_R^{k-1}}{(k+1)!} \right),
\]

where \( \chi_{i,L} \) and \( \chi_{i,R} \) are the \( i \)th Chern characters at the node on the genus-1 com-
ponent and the genus-\((g - 1)\) component, respectively.

This proposition highlights that the Chern character splits along \(D\) leaving us to just multiply the coefficient of \(J_k\).

**Proposition 3.4. [12]**

\[
J_i J_k = D \left( -\psi_L - \psi_R \right) \left( \frac{\psi_R^i - \psi_L \psi_R^{i-1}}{(i + 1)!} \right) \left( \frac{\psi_R^k - \psi_L \psi_R^{k-1}}{(k + 1)!} \right)
\]

Here the proposition says we need to multiply the coefficients of \(J_i\) and \(J_k\) while adding \((-\psi_L - \psi_R\)). We are now able to use Macaulay2 to compute our intersection numbers

\[
\int_{\tilde{M}_{g,1}} \tilde{\lambda}_{g-i} \psi_1^{2g-2+i}.
\]

We compute these values and bring back the values from the intersection numbers on \(\overline{M}_{g,1}\). The tables suggest that integrating over \(\overline{M}_{g,1}\) and \(\tilde{M}_{g,1}\) is the same leading us to our conjecture.

**Table 3.1: Intersection Numbers over \(\overline{M}_{g,1}\)**

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<th>3</th>
<th>4</th>
<th>5</th>
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<tr>
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<td>1/106168320</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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</table>
Table 3.2: Intersection Numbers over $\tilde{M}_{g,1}$

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<td>0</td>
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</tbody>
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3.2 Main Results

With the tables in the previous section suggesting the intersection numbers on both compactifications should be the same, we have our conjecture.

**Conjecture 3.1.** Let $g > 1$.

$$\int_{\tilde{M}_{g,1}} \tilde{\lambda}_{g-i} \tilde{\psi}_1^{2g-2+i} = \int_{\tilde{M}_{g,1}} \lambda_{g-i} \psi_1^{2g-2+i}$$

By first recognizing along the diagonals we were using the same $\tilde{\lambda}$-class, it might be best to take that approach in proving for $\tilde{\lambda}_{g-i}$ case for when the choice for $g$ and $i$ are close. This leads us to proving our main theorem, which is a special case of our conjecture.

**Theorem 3.5.** Let $g > 1$.

$$\int_{\tilde{M}_{g,1}} \tilde{\lambda}_{g-i} \tilde{\psi}_1^{2g-2+i} = \int_{\tilde{M}_{g,1}} \lambda_{g-i} \psi_1^{2g-2+i}$$
for $g \leq i + 3$.

**Proof.** We prove this in cases:

- **$g = i$:** We want to show $\int_{\tilde{M}_{g,1}} \psi_1^{3g-2} = \int_{M_{g,1}} \psi_1^{3g-2}$, which is true because of Proposition 3.1.

- **$g = 1 + i$:** We want to show $\int_{\tilde{M}_{g,1}} \tilde{\lambda}_1 \psi_1^{3g-3} = \int_{M_{g,1}} \lambda_1 \psi_1^{3g-3}$. Remember,

  \[
  \tilde{\lambda}_1 = \tilde{c}_1 = \tilde{c}_1 = \lambda_1 = \lambda_1 + J_1.
  \]

  After distributing $\psi_1^{3g-3}$, it follows $\int_{M_{g,1}} J_1 \psi_1^{3g-3} = 0$ because

  \[
  \int_{M_{g,1}} J_1 \psi_1^{3g-3} = \int_{M_{g,1}} D \psi_1^{3g-3}
  \]

  where the irreducible genus 1 component of $D$ has no $\psi$-class. Thus we have our desired result.

- **$g = 2 + i$:** We want show $\int_{\tilde{M}_{g,1}} \tilde{\lambda}_2 \psi_1^{3g-4} = \int_{M_{g,1}} \lambda_2 \psi_1^{3g-4}$. Note:

  \[
  \tilde{\lambda}_2 = \lambda_2 + ch_1 J_1 + \frac{1}{2} J_1^2 + J_2
  \]

  where we need the last three terms to vanish. After distributing $\psi_1^{3g-4}$, we
notice

\[ \int_{M_{g,1}} ch_1 J_1 \psi_1^{3g-4} = \int_{M_{g,1}} D(ch_{1,L} + ch_{1,R}) \psi_1^{3g-4} = \int_{M_{g,1}} Dch_{1,L} \psi_1^{3g-4} \]

because there is no Chern character on the irreducible genus 1 component. Moving on to the next term,

\[ \int_{M_{g,1}} \frac{1}{2} J_1^2 \psi_1^{3g-4} = \int_{M_{g,1}} -\frac{1}{2} D(\psi_L + \psi_R) \psi_1^{3g-4} = \int_{M_{g,1}} -\frac{1}{2} D\psi_L \psi_1^{3g-4} \]

because there is no \( \psi \)-class on the irreducible genus 1 component. And the last term,

\[ \int_{M_{g,1}} J_2 \psi_1^{3g-4} = \int_{M_{g,1}} -\frac{1}{2} D(\psi_L + \psi_R) \psi_1^{3g-4} = \int_{M_{g,1}} -\frac{1}{2} D\psi_L \psi_1^{3g-4} \]

for the same reasoning. Using the fact \( \int_{M_{1,1}} \psi_L = \int_{M_{1,1}} ch_{1,L} \).
\[
\int_{M_{g,1}} D\chi_{1,L} \psi_1^{3g-4} - \frac{1}{2} \int_{M_{g,1}} D\psi_L \psi_1^{3g-4} - \frac{1}{2} \int_{M_{g,1}} D\psi_L \psi_1^{3g-4}
\]
\[
= \int_{M_{g,1}} D\chi_{1,L} \psi_1^{3g-4} - \int_{M_{g,1}} D\psi_L \psi_1^{3g-4}
\]
\[
= \int_{M_{1,1}} \chi_{1,L} \int_{M_{g-1,2}} \psi_1^{3g-4} - \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_1^{3g-4}
\]
\[
= \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_1^{3g-4} - \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_1^{3g-4}
\]
\[
= 0
\]

Thus we have our desired result.

- \(g = 3 + i\): We want to show \(\int_{M_{g,1}} \tilde{\lambda}_3 \psi_1^{3g-5} = \int_{M_{g,1}} \lambda_3 \psi_1^{3g-5}\). We want to remember

\[
\tilde{\lambda}_3 = \lambda_3 + 2J_3 + \frac{1}{2}c_1J_1 + \frac{1}{2}c_1J_2 + \frac{1}{6}J_1^3 + c_1J_2 + ch_1J_1 + J_1J_2
\]

where we want the remaining terms to cancel. After distributing \(\psi_1^{3g-5}\), we have

\[
\int_{M_{g,1}} 2J_3 \psi_1^{3g-5} = \int_{M_{g,1}} \frac{1}{3} D(p_{R}^2 - \psi_L \psi_R) \psi_1^{3g-5}
\]
\[
= \int_{M_{g,1}} \frac{-1}{3} \psi_L \psi_R \psi_1^{3g-5}
\]
\[
= \frac{-1}{3} \int_{M_{1,1}} \psi_L \int_{M_{g-1,1}} \psi_R \psi_1^{3g-5}
\]
because there is no $\psi$-class on the irreducible genus 1 component. Also,

$$\int_{\overline{M}_{g,1}} \frac{1}{2} ch_1^2 J_1 \psi_1^{3g-5} = \int_{\overline{M}_{g,1}} \frac{1}{2} \left( ch_{1,L}^2 + 2 ch_{1,L} ch_{1,R} + ch_{1,R}^2 \right) \psi_1^{3g-5}$$

$$= \int_{\overline{M}_{g,1}} \frac{1}{2} D \left( ch_{1,L}^2 + 2 ch_{1,L} ch_{1,R} \right) \psi_1^{3g-5}$$

because there is no Chern character on the irreducible genus 1 component.

We can simplify this more by

$$\int_{\overline{M}_{g,1}} \frac{1}{2} D \left( ch_{1,L}^2 + 2 ch_{1,L} ch_{1,R} \right) \psi_1^{3g-5} = \int_{\overline{M}_{g,1}} \frac{1}{2} D ch_{1,L}^2 \psi_1^{3g-5}$$

$$+ \int_{\overline{M}_{g,1}} D ch_{1,L} ch_{1,R} \psi_1^{3g-5}$$

where $ch_{1,L}^2 = \psi_L^2 = 0$ on $\overline{M}_{1,1}$. This implies

$$\int_{\overline{M}_{g,1}} \frac{1}{2} D ch_{1,L}^2 \psi_1^{3g-5} + \int_{\overline{M}_{g,1}} ch_{1,L} ch_{1,R} \psi_1^{3g-5} = \int_{\overline{M}_{g,1}} D ch_{1,L} ch_{1,R} \psi_1^{3g-5}$$

giving us

$$\int_{\overline{M}_{g,1}} \frac{1}{2} ch_1^2 J_1 \psi_1^{3g-5} = \int_{\overline{M}_{g,1}} D ch_{1,L} ch_{1,R} \psi_1^{3g-5} = \int_{\overline{M}_{1,1}} \psi_L \int_{\overline{M}_{g-1,1}} ch_1 \psi_1^{3g-5}$$

because $ch_{1,L} = \psi_L$ on $\overline{M}_{1,1}$. Next,
\[ \int_{\mathcal{M}_{g,1}} \frac{1}{2} ch_1 J_2^2 \psi_1^{3g-5} = \int_{\mathcal{M}_{g,1}} \frac{-1}{2} (ch_{1,L} + ch_{1,R}) (\psi_L + \psi_R) \psi_1^{3g-5} \]
\[= \int_{\mathcal{M}_{g,1}} \frac{-1}{2} (ch_{1,L} \psi_R + ch_{1,R} \psi_L) \psi_1^{3g-5} \]

because there is no \( \psi \)-class or Chern character on the irreducible genus 1 component. For similar reason as before we get

\[ \int_{\mathcal{M}_{g,1}} \frac{1}{2} ch_1 J_2^2 \psi_1^{3g-5} = \frac{-1}{2} \int_{\mathcal{M}_{1,1}} \psi_L \int_{\mathcal{M}_{g-1,2}} \psi_R \psi_1^{3g-5} - \frac{1}{2} \int_{\mathcal{M}_{1,1}} \psi_L \int_{\mathcal{M}_{g-1,2}} ch_1 \psi_1^{3g-5}. \]

Moving onto the next term,

\[ \int_{\mathcal{M}_{g,1}} \frac{1}{6} J_1^3 \psi_1^{3g-5} = \int_{\mathcal{M}_{g,1}} \frac{1}{6} (\psi_L^2 + 2 \psi_L \psi_R + \psi_R^2) \psi_1^{3g-5} = \int_{\mathcal{M}_{g,1}} \frac{1}{3} \psi_L \psi_R \psi_1^{3g-5} \]

because \( \psi_L^2 = 0 \) over \( \mathcal{M}_{1,1} \) and there is no \( \psi \)-class on the irreducible genus 1 component. Thus

\[ \int_{\mathcal{M}_{g,1}} \frac{1}{6} J_1^3 \psi_1^{3g-5} = \frac{1}{3} \int_{\mathcal{M}_{1,1}} \psi_L \int_{\mathcal{M}_{g-1,2}} \psi_R \psi_1^{3g-5}. \]
Finally,

\[
\int_{\bar{M}_{g,1}} ch_1 J_2 \psi_1^{3g-5} = \int_{\bar{M}_{g,1}} \frac{1}{2} D (ch_{1,L} + ch_{1,R}) (\psi_R - \psi_L) \psi_1^{3g-5} \\
= \int_{\bar{M}_{g,1}} \frac{1}{2} D (ch_{1,L} \psi_R - ch_{1,R} \psi_L) \psi_1^{3g-5}
\]

because there is no \(\psi\)-class or Chern character on the irreducible genus 1 component. It follows

\[
\int_{\bar{M}_{g,1}} \frac{1}{2} D (ch_{1,L} \psi_R - ch_{1,R} \psi_L) \psi_1^{3g-5} = \frac{1}{2} \int_{\bar{M}_{g,1}} Dch_{1,L} \psi_R \psi_1^{3g-5} \\
- \frac{1}{2} \int_{\bar{M}_{g,1}} Dch_{1,R} \psi_L \psi_1^{3g-5} \\
= \frac{1}{2} \int_{\bar{M}_{1,1}} ch_{1,L} \int_{\bar{M}_{g-1,2}} \psi_R \psi_1^{3g-5} \\
- \frac{1}{2} \int_{\bar{M}_{1,1}} \psi_L \int_{\bar{M}_{g-1,2}} ch_{1,R} \psi_1^{3g-5}
\]

where we finally use the fact \(ch_{1,L} = \psi_L\) on \(\bar{M}_{1,1}\) to say

\[
\int_{\bar{M}_{g,1}} ch_1 J_2 \psi_1^{3g-5} = \frac{1}{2} \int_{\bar{M}_{1,1}} \psi_L \int_{\bar{M}_{g-1,2}} \psi_R \psi_1^{3g-5} - \frac{1}{2} \int_{\bar{M}_{1,1}} \psi_L \int_{\bar{M}_{g-1,2}} ch_{1,R} \psi_1^{3g-5}.
\]
Notice that the last term vanishes because

$$
\int_{M_{g,1}} J_1 J_2 \psi_1^{3g-5} = \int_{M_{g,1}} \frac{-1}{2} D (\psi_L + \psi_R) (\psi_R - \psi_L) \psi_1^{3g-5}
$$

$$
= \int_{M_{g,1}} \frac{-1}{2} D (\psi_R^2 - \psi_L^2) \psi_1^{3g-5}
$$

$$
= 0
$$

because $\psi_L^2 = 0$ over $\overline{M}_{1,1}$ and there is no $\psi$-class on the irreducible genus 1 component. Putting everything together

$$
\frac{-1}{3} \int_{M_{1,1}} \psi_L \int_{M_{g-1,1}} \psi_R \psi_1^{3g-5} + \int_{M_{1,1}} \psi_L \int_{M_{g-1,1}} \psi_L \int_{M_{g-1,1}} \psi_1^{3g-5} + \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_R \psi_1^{3g-5}
$$

$$
- \frac{1}{2} \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_R \psi_1^{3g-5}
$$

$$
- \frac{1}{2} \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_1^{3g-5}
$$

$$
+ \frac{1}{3} \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_R \psi_1^{3g-5}
$$

$$
+ \frac{1}{2} \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_1^{3g-5}
$$

$$
+ \frac{1}{2} \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_R \psi_1^{3g-5}
$$

$$
- \frac{1}{2} \int_{M_{1,1}} \psi_L \int_{M_{g-1,2}} \psi_1^{3g-5}
$$

$$
= 0
$$

giving us our desired result.
Bibliography


