Abstract. This paper presents the Vandermonde matrix and considers the connection between the Vandermonde matrix and the cyclic polytope. The matrix of the set of vertices of a cyclic polytope form a Vandermonde matrix. Using the Vandermonde determinant a proof for the affine independence of any \(d + 1\) vertices of the cyclic polytope in \(\mathbb{R}^d\) will be presented. A proof that the cyclic polytope is simplical will also be presented.

1. Introduction

The Vandermonde matrix and the Vandermonde determinant are named after the French musician Alexandre Théophile Vandermonde (1735-1796). Vandermonde did not begin his mathematical career until the age of 35 and only published four papers. He is best known for his work on determinant theory. In his last paper, Vandermonde showed that a matrix with two identical rows or columns will have determinant equal to zero. Surprisingly, the Vandermonde did not write about the Vandermonde determinant in any of his papers. The famous French mathematician, Henri Lebesgue, believed that someone had misread Vandermonde’s notations which led to this misnaming. [1]

2. Vandermonde Matrix

Definition 1. The Vandermonde matrix is an \(m \times n\) matrix defined by

\[
V = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & a_1 & a_2 & \cdots & a_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \cdots & a_m^{n-1}
\end{bmatrix}
\]

where the entries \(\alpha_{ij} = a_i^{j-1}\).

In this paper, we will discuss an application of the square \(n \times n\) Vandermonde matrix to cyclic polytopes. There are a few important properties of the square Vandermodne matrix that will be used later.

Theorem 1. The Vandermonde determinant is

\[
\det(V) = \prod_{0 \leq i < j \leq n} (a_j - a_i)
\]

Proof. By Leibniz formula,

\[
\det(V) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_i^{\sigma(1)-1} a_2^{\sigma(2)-1} \cdots a_n^{\sigma(n)-1}
\]

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where $S_n$ is the set of permutations of \( \{1, 2, \ldots, n\} \) and $sgn(\sigma)$ is either $-1$ or $1$ for odd or even permutations, respectively. Clearly, $\det(V)$ is a polynomial in $a_0, a_1, \ldots, a_n$. Each term of $\det(V)$ is the product of an element from every row of $V$, i.e. an element of degree 0, an element of degree 1, and so on until the last row with an element of degree $n - 1$. Since

$$0 + 1 + 2 + \cdots + (n - 1) = \frac{n(n - 1)}{2} = \binom{n}{2}$$

So each term will be of degree $\frac{n(n-1)}{2}$. Hence, $\det(V)$ is a homogenous polynomial.

Let $f(a_0, a_1, \ldots, a_n) = \det(V)$. For any $0 \leq i < j \leq n$, if $a_i = a_j$ then $V$ will have two identical columns, so $\det(V) = 0$. Since the determinant is a polynomial and $a_i = a_j$ is a root of $f$, $(a_j - a_i)$ is a factor of $f$.

Let, $g(a_0, a_1, \ldots, a_n) = \prod_{0 \leq i < j \leq n}(a_j - a_i)$. Since there are $\binom{n}{2}$ choices for $i, j$ where $0 \leq i < j \leq n$, $\deg(g) = \binom{n}{2}$. Hence, $f$ and $g$ are both homogeneous with the same degree and have the same factorization. So, $f = c \cdot g$ for some constant $c \in \mathbb{R}$. However, the coefficient of the $a_0^0a_1^2 \cdots a_n^d$ term of $f$ and $g$ is 1. So $c = 1$ and $f = g$.

\[\square\]

**Corollary 2.** The Vandermonde matrix is invertible if and only if $a_1, a_2, \ldots, a_n$ are distinct. Consequently, if $a_0, a_1, \ldots, a_n$ are distinct then the determinant of $V$ is nonzero. [2]

### 3. Cyclic Polytopes

Vandermonde matrices have many applications such as polynomial interpolation and Fourier transformations. Here we will discuss the applications of the Vandermonde matrix to cyclic polytopes. A proof for the affine independence of $d + 1$ vertices of the $d$-dimensional cyclic polytope can be beautifully and simply presented using the Vandermonde matrix. As a result of the affine independence of $d + 1$ vertices we can show that the cyclic polytope is simplicial.

**Definition 2.** Let $V$ be a set of points in $\mathbb{R}^d$. $V$ is convex if for any two points $v, u \in V$, there exists a line segment $L$, between $v, u$ is also contained in $V$. The convex hull of $V$, $\text{conv}(V)$, is the smallest convex set containing $V$. [4]

**Definition 3.** A polytope is a set of points $P$ in $\mathbb{R}^d$ where $P$ is the convex hull of finitely many points in $\mathbb{R}^d$. [4]

Let $n, d \in \mathbb{N}$. Consider the map $x : \mathbb{R} \to \mathbb{R}^d$ where for all $v \in \mathbb{R}$,

$$x(v) = \begin{bmatrix} v \\ v^2 \\ \vdots \\ v^d \end{bmatrix}$$

The moment curve is $\{x(v) : v \in \mathbb{R}\}$.

**Definition 4.** For any $n > d$, the cyclic polytope, $C_d(n)$, is the convex hull of $n$ distinct points $\{x(v_1), x(v_2), \ldots, x(v_n)\}$ on the moment curve where $v_1 < v_2 < \cdots < v_n$. [3]

$$C_d(n) = \text{conv}(x(v_1), x(v_2), \ldots, x(v_n))$$

**Example 1.** In $\mathbb{R}^2$ the moment curve is

$$\left\{x(v) = \begin{bmatrix} v \\ v^2 \end{bmatrix} : v \in \mathbb{R}\right\}$$
The moment curve is given by \( y = x^2 \).

Let \( \{x(v_1), x(v_2), x(v_3), x(v_4), x(v_5)\} \) be a set of five distinct points on \( y = x^2 \) where \( v_1 < v_2 < v_3 < v_4 < v_5 \) and \( v_i \in \mathbb{R} \) for \( 1 \leq i \leq 5 \). Let,

\[
C_2(5) = \text{conv}(x(v_1), x(v_2), x(v_3), x(v_4), x(v_5))
\]

\( C_2(5) \) is pictured in Figure 1.

(\( A \)) \( C_2(5) \)       (\( B \)) \( C'_2(5) \)

**Figure 1**

\( C_2(5) \) is a two dimensional polytope with five vertices and five edges. Consider another set of five distinct points on \( y = x^2 \), \( \{x(w_1), x(w_2), x(w_3), x(w_4), x(w_5)\} \) where \( w_1 < w_2 < w_3 < w_4 < w_5 \) and \( w_i \in \mathbb{R} \) for \( 1 \leq i \leq 5 \). Let,

\[
C'_2(5) = \text{conv}(x(w_1), x(w_2), x(w_3), x(w_4), x(w_5))
\]

\( C'_2(5) \) is also pictured in Figure 1. By labeling the vertices, a bijection can be defined between the vertices, edges, and the whole polytope of \( C_2(5) \) and \( C'_2(5) \).

**Proposition 3.** Two polytope \( P, Q \) are combinatorially equivalent if there exists a bijection between the face of \( P \) and the faces of \( Q \) that preserves inclusion relation. [4]

Although \( C_2(5) \) and \( C_2(5) \) are the convex hulls of different points on \( y = x^2 \) and have different side lengths, they both belong in the same combinatorial equivalence class. The combinatorial structure of \( C(n, d) \) is independent of the choices of the \( n \) points. Thus we can discuss the cyclic polytope by using only the dimension of the polytope and the number of points we choose on the moment curve.

**Definition 5.** A set of vectors \( \{v_1, v_2, \ldots, v_k\} \) are affinely independent if \( \sum_{i=1}^{k} \lambda_i = 0, \sum_{i=1}^{k} \lambda_i v_i = 0 \) implies that \( \lambda_i = 0 \) for all \( i \).

**Theorem 4.** Any \( d + 1 \) vertices of \( C_d(n) \) are affinely independent.
Proof. For $n > d > 2$, let $\{v_1, v_2, \ldots, v_n\}$ be $n$ distinct points on the moment curve. Pick any $d + 1$ points from the set $\{v_1, v_2, \ldots, v_n\}$. Now consider the Vandermonde matrix $V$ where

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x(v_1) & x(v_2) & \cdots & x(v_d) \end{bmatrix}$$

Since $V$ is a square Vandermonde matrix,

$$\det(V) = \prod_{1 \leq i < j \leq d} (v_j - v_i)$$

Consider the columns vectors of $V$.

$$\begin{bmatrix} 1 \\ x(v_1) \end{bmatrix}, \begin{bmatrix} 1 \\ x(v_2) \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ x(v_d) \end{bmatrix}$$

By Theorem 2, since $\{v_1, v_2, \ldots, v_n\}$ are $n$ distinct points the determinant of $V$ is nonzero. Since the determinant of $V$ is nonzero the columns vectors of linearly independent. Hence,

$$a_1 \begin{bmatrix} 1 \\ x(v_1) \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ x(v_2) \end{bmatrix} + \cdots + a_d \begin{bmatrix} 1 \\ x(v_d) \end{bmatrix} = 0$$

when $a_i = 0$ for all $i$.

Hence,

$$\sum_{i=1}^{d} a_i = 0$$

and

$$\sum_{i=1}^{d} a_i x(v_i) = 0$$

Suppose $a_i \neq 0$ for some $i$. But this contradicts the linearly independence of the columns of $V$. Hence, $a_i = 0$ for all $i$. By Definition 5, $x(v_1), x(v_2), \ldots, x(v_n)$ are affinely independent. Therefore, any $d + 1$ vertices of $C_d(n)$ are affinely independent.

\[\square\]

**Definition 6.** A $d$-polytope is simplicial if every facet of the polytope is a simplex. [4] A simplex is a $n$-dimensional polytope with only $n + 1$ vertices.

**Definition 7.** Let $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$. The $\sum_{i=1}^{m} \lambda_i v_i$ where $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$ is an affine combination. [4]

**Definition 8.** Let $M$ be a set of $m \geq 0$ of points in $\mathbb{R}^d$. The affine hull of $M$ is the set of all affine combinations of points from $M$. The dimension of a polytope is the dimension of its affine hull. [4]

**Lemma 5.** Let $A$ be an affinely independent set with $m \geq 0$ points in $\mathbb{R}$. The affine hull of $A$ has dimension $m - 1$. [4]

**Proof.** Let $U = \{u_1, u_2, \ldots, u_m\}$ be an affinely independent set of $m \geq 0$ points in $\mathbb{R}^n$. Consider,

$$\begin{bmatrix} 1 \\ u_1 \\ 1 \\ u_2 \\ \cdots \\ 1 \\ u_m \end{bmatrix}$$

By Definition 5, $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i u_i = 0$ implies that $\lambda_i = 0$ for all $i$. Hence,

$$\lambda_1 \begin{bmatrix} 1 \\ u_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ u_2 \end{bmatrix} + \cdots + \lambda_m \begin{bmatrix} 1 \\ u_m \end{bmatrix} = 0$$
when $\lambda_i = 0$ for all $i$. So, these vectors are linearly independent. Let $S$ be the span of these vectors. So $S$ is a $m$-dimensional linear space in $\mathbb{R}^{n+1}$. Let $u \in \mathbb{R}^{n+1}$. And let $u$ be an affine combination of $u_i$. By Definition 7, $u = \sum_{i=1}^{m} \lambda_i u_i$ where $\lambda_i \leq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$. Which implies that $u \in S$. $u \in S \cap \{(1, x) \in \mathbb{R}^{n+1}\}$. By Definition 8, $u$ is in the affine hull of $\{u_1, u_2, \ldots, u_m\}$. Hence, the affine hull of $\{u_1, u_2, \ldots, u_m\}$ has dimension $m - 1$.

\[ \square \]

**Theorem 6.** $C_d(n)$ is simplicial.

**Proof.** To prove Theorem 6, we need to show that every facet, $(d - 1)$-dimensional face, of $C_d(n)$ is the convex hull of only $d$ vertices. Recall Example 1. $C_2(5)$ is a 2-dimensional polytope. A facet of $C_2(5)$ is a 1-dimensional face, i.e. an edge. It is clear from the picture that any facet of $C_2(5)$ is the convex hull of exactly 2 vertices, and therefore a simplex.

For $n > d$, let $C_d(n) = \text{conv}(v_1, v_2, \ldots, v_n)$ where $\{v_1, v_2, \ldots, v_n\}$ are $n$ distinct points on the moment curve. Consider an facet, $F$, in $C_d(n)$. Since $F$ is a $(d - 1)$-dimensional polytope $F$ is the convex hull of at least $d$ vertices.

Suppose $F$ has more than $d$ vertices. Let $F$ be the convex hull of $d + 1$ vertices. By Corollary 1, any $d + 1$ vertices of $C_d(n)$ are affinely independent. $F$ has dimension $(d - 1)$ so the dimension of its affine hull is also $(d - 1)$. Corollary 1 implies that the vertices of $F$ are affinely independent. Since the dimension of a polytope is the dimension of its affine hull [4]. However, by Lemma 5, affine independence implies that the vertices of $F$ have an affine hull of dimension $d$. And then $F$ would actually be a $d$-polytope. Contradiction. Hence, $F$ has exactly $d$ vertices. Therefore, every facet of $C_d(n)$ is a simplex. Thus, $C_d(n)$ is simplicial.

\[ \square \]

**References**