CONICS ON THE PROJECTIVE PLANE

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ABSTRACT. In this paper, we discuss a special property of conics on the projective plane and answer questions in enumerative algebraic geometry such as "How many points determine a conic?" and "How many conics do we expect to pass through \( m \) points and tangent to \( n \) lines?"

1. Introduction

Conics have been studied since the time of Menaechmus and Euclid. One of the earliest known work on the subject involves counting the circles tangent to three non-degenerate circles, now known as the Apollonius’ Circles Problem. Aspects of this problem have inspired generations of mathematicians in the 19th century and led to a number of interesting discoveries [1]. However, it was not until the development of superstring theory in the 1980s that problems such as counting rational curves on projective spaces were better understood (for more information, see [2], [3]).

In this paper, we introduce the concept of a projective space and define the notion of a variety. We discuss the property that all conics in the complex projective space are projectively equivalent. In the last section, we introduce a useful concept, Bezout’s theorem, that answers many questions in enumerative algebraic geometry such as the number of points that determine a conic and the number of conics passing through \( m \) points and tangent to \( n \) lines.

2. Preliminaries

The machinery behind this exposition includes ideals in \( \mathbb{C}[x_1, \ldots, x_n] \) and affine algebraic varieties in \( \mathbb{C}^n \). Maps between these structures provide a rich interplay between algebraic techniques and geometric concepts. We define an affine variety \( V \) and the ideal that vanish on \( V \) as follows.

Definition 1. An affine variety is the zero-locus of a collection of polynomials \( F = \{F_i\}_{i \in I} \subseteq \mathbb{C}[x_1, \ldots, x_n] \). That is

\[
V = \mathbb{V}(F) = \{a \in \mathbb{C}^n : F_i(a) = 0, \ \forall i \in I\}
\]

Definition 2. Given \( V \subseteq \mathbb{C}^n \), the ideal of polynomials that vanish on \( V \) is defined

\[
\mathcal{I}(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(v) = 0, \ \forall v \in V\}
\]

Example 1. Consider the twisted cubic curve, \( V = \mathbb{V}(x^2 - y, x^3 - z) = \mathbb{V}(x^2 - y) \cap \mathbb{V}(x^3 - z) \). Let \( p = (t, t^2, t^3) \) where \( t \in \mathbb{C} \). By substitution, we have that \( x^2 - y = t^2 - y = 0 \) and \( x^3 - z = t^3 - t^3 = 0 \). Hence, \( V = \{(t, t^2, t^3) : t \in \mathbb{C}\} \).

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Note that the complements of algebraic varieties define a topology in \( \mathbb{C}^n \) called the Zariski Topology \([4]\). Hilbert’s Nullstellensatz show a one-to-one correspondence between radical ideals \( \mathcal{I}(V) \) and the Zariski closure \( \mathcal{V}(\mathcal{I}(V)) \subset \mathbb{C}^n \) \([4]\).

The above framework presents some difficulty when proving statements involving intersections without the need to consider different cases separately. As an example, suppose \( l_1 \) and \( l_2 \) are two distinct lines in \( \mathbb{C}^2 \). By the parallel postulate in Euclidean geometry, either the two lines meet at a point or they do not. We also have the additional case where non-distinct lines intersect at infinitely many points. Therefore, it is useful to take the projective closure of \( \mathbb{C}^2 \) by including points at infinity. We now discuss the \( n \)-dimensional projective space \( \mathbb{C}P^n \) or \( \mathbb{P}^n \).

**Definition 3.** The Projective \( n \)-space \( \mathbb{P}^n \) is the set of all lines through the origin in \( \mathbb{C}^{n+1} \).

Let \( \sim \) be an equivalence relation of points in the same line that passes through the origin where \((x_0, \cdots, x_n) \sim (x_0', \cdots, x_n')\) if and only if \((x_0, \cdots, x_n) = \lambda(x_0', \cdots, x_n')\) for some \( \lambda \neq 0 \in \mathbb{C} \). We see that \( \mathbb{P}^n = \mathbb{C}^{n+1}\backslash\{0\} \).

**Definition 4.** homogeneous coordinates \([x_0 : \cdots : x_n] \in \mathbb{P}^n \) is the equivalence class \( \{\lambda(x_0, \cdots, x_n) : \lambda \in \mathbb{C}\} \).

**Example 2.** Consider the following points in \( \mathbb{C}^3 : (2, 6, 10), (\frac{2}{3}, \frac{4}{3}, \frac{10}{3}), (1+i, 2+2i, 5+5i) \). Note that \( 2(1, 2, 5) = (2, 6, 10), \frac{2}{3}(1, 2, 5) = (\frac{2}{3}, \frac{4}{3}, \frac{10}{3}) \), and \((1+i)(1, 2, 5) = (1+i, 2+2i, 5+5i) \). Thus, the three points lie on the same line in \( \mathbb{C}^3 \) and define the point \([1 : 2 : 5] \in \mathbb{P}^2 \).

Consider the projective line \( \mathbb{P}^1 \) and let \( x_0 = 1 \) be a complex reference line. Note that for any given line in \( \mathbb{C}^2 \) passing through the origin, we can identify a point in \( \mathbb{C} \) with the bijective mapping \( \psi_0 : \mathbb{P}^1 \to \mathbb{C} \) defined by \([1 : x_1] \mapsto x_1 \). When \( x_0 = 0 \), no point can be identified since the line through the origin is parallel to the reference line. We refer to this point as a point at infinity. Therefore, \( \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \).

Similarly, consider the projective plane \( \mathbb{P}^2 \) and let \( x_0 = 1 \) be a complex reference plane not passing through the origin. Note that for any given line in \( \mathbb{C}^3 \) passing through the origin, we can identify a point \((x_1, x_2) \in \mathbb{C}^2 \). The points at infinity in this case are lines in \( \mathbb{C}^2 \) that are parallel to reference plane. However, this is just a copy of \( \mathbb{P}^1 \). It follows that \( \mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{C} \cup \{\infty\} = \mathbb{C}^2 \cup \mathbb{P}^1 \). Extending this construction to an \( n \)-dimensional projective space, we obtain \( \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1} \) \([5]\).

Let us see if we can adapt the notion of an affine variety to projective spaces. Consider \( V = \mathcal{V}(y^3 - x^2) \subset \mathbb{P}^2 \). Note that \((1, 1, 1) \in V \). Herein lies the problem. Since \((2, 2, 2) = 2(1, 1, 1) \) in \( \mathbb{P}^2 \), \((2, 2, 2) \) must be in \( V \). But \( 2^3 - 2^2 \neq 0 \). We use homogeneous polynomials to work around this problem. Homogenizing \( y^3 - x^2 \) gives \( y^3 - x^2z \). Since \( 2^3 - 2^2 \cdot 2 = 0 \), it follows that \((2, 2, 2) \in \mathcal{V}(y^3 - x^2z) \). We now define a projective variety. Henceforth, varieties are assumed to be projective varieties.

**Definition 5.** A projective variety in \( \mathbb{P}^n \) is the zero-locus of \( \{F_i\}_{i \in I} \) of homogeneous polynomials in \( n + 1 \) variables.

To conclude this section, we return to the case of two lines in \( \mathbb{C}^2 \) and find their intersection after taking their projective closure.
Example 3. Let \( l_1 \) and \( l_2 \) be parallel lines in \( \mathbb{C}^2 \) defined by the equations \( Y = aX + b \) and \( Y = aX + c \) respectively. Homogenizing the equations yields \( Y = aX + bZ \) and \( Y = aX + cZ \). It is clear that \( l_1 \) and \( l_2 \) intersect when \( [1 : a : 0] \). Suppose \( l_1 \) and \( l_2 \) are non-distinct lines instead. Although they intersect at infinitely many points in \( \mathbb{C}^2 \), all of those points belong in the same equivalence class in \( \mathbb{P}^2 \). Hence, we have a unique intersection for all three cases.

3. Conics on the Projective Plane

We obtain many interesting results by taking the projective closure of conic sections in \( \mathbb{C}^2 \). Recall that a conic in \( \mathbb{C}^2 \) is the affine algebraic variety

\[
V(aX^2 + bXY + cY^2 + dX + eY + f) \subset \mathbb{C}^2
\]

Up to a linear change of coordinates, we can show that any irreducible quadratic polynomial \( F \in \mathbb{C}[x,y] \) is either a parabola \( V(Y - X^2) \) or a hyperbola \( V(YX - 1) \).

Theorem 1. Up to an affine change of coordinates in the affine plane, there exists exactly two isomorphic affine conic plane curves. (Exercise 3.5.3 in [4])

It turns out that if we take the projective closure in \( \mathbb{P}^2 \) of the conic in (3.1)

\[
V(aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2) \subset \mathbb{P}^2
\]

we obtain a much stronger result, that all non-degenerate conics in \( \mathbb{P}^2 \) are projectively equivalent.

We begin by stating a lemma, which we will not prove. A proof for the general case of a quadric in \( k[x_0, \ldots, x_n] \) where \( \text{char}(k) \neq 2 \) can be found in [5].

Lemma 1. Let \( f = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 \in \mathbb{C}[X,Y,Z] \) be a non-zero polynomial. Then \( V(f) \) is projectively equivalent to a quadric defined by an equation of the form

\[
c_0X^2 + c_1Y^2 + c_2Z^2 = 0
\]

where \( c_0, c_1, c_2 \in \mathbb{C} \) not all zero.

The main idea is to find the rank of a conic and be able to say that all conics of such rank are projectively equivalent. We define the rank of a conic as follows.

Definition 6. Let \( C \subset \mathbb{P}^2 \) be a conic. \( C \) has a rank equal to the number of non-zero \( c_i \)'s if \( C \) is projectively equivalent to the conic defined by the equation in (3.3).

In particular, the rank of \( X^2 - Y^2 + Z^2 = 0 \), \( X^2 - Z^2 = 0 \) and \( X^2 = 0 \) are 3, 2, and 1 respectively.

Note that the defining equation for the variety in (3.2) can be expressed as

\[
f(x) = x^T M x
\]

where

\[
M = \begin{bmatrix}
a & b/2 & d \\
 b/2 & c & e/2 \\
 d/2 & e/2 & f \\
\end{bmatrix}
\]
is symmetric and $x = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$.

The following lemma shows how to find the rank of $\mathbb{V}(f)$ by looking at the matrix $M$.

**Lemma 2.** Let $f = x^T M x$ where $M$ is a symmetric $3 \times 3$ matrix.

1. Given an $A \in \text{GL}_3$, let $B = A^{-1}$. Then
   \[ A(\mathbb{V}(f)) = \mathbb{V}(g). \]
   
   where $g(x) = x^T B^T M B x$.

2. The rank of a conic $\mathbb{V}(f)$ equals the rank of $M$.

**Proof.** We can easily check that $A(\mathbb{V}(f)) = \mathbb{V}(g)$ where $g = f \circ B$. Then
   \[ g(x) = f(Bx) = (Bx)^T M (Bx) = x^T B^T M B x \]

   By Lemma 1 and Lemma 2 pt.(1), there exists a $3 \times 3$ matrix in $\text{GL}_3$ such that $g = c_0 X^2 + c_1 Y^2 + c_2 Z^2$ where $c_0, c_1, c_2$ not all zero and so
   \[ g(x) = x^T B^T M B x = x^T \begin{bmatrix} c_0 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{bmatrix} x \]

   where $B = A^{-1}$. Since $\text{rank}(B^T M B) = \#(\text{non-zero } c_i)$, $\text{rank}(M) = \#(\text{non-zero } c_i)$.

When $\text{rank}(M) = 3$, the conic is non-degenerate. That is, it is not the empty set, two lines, a double line, or a point. We can now prove the statement in the beginning of the section.

**Theorem 2.** Any two conics in $\mathbb{CP}^2$ are projectively equivalent if and only if they have the same rank.

**Proof.** By Lemma 1, any conic is projectively equivalent to $c_0 X^2 + c_1 Y^2 + c_2 Z^2 = 0$ where not all $c_i$’s are zero. Since $\mathbb{C}$ is algebraically closed, $x^2 - c_i$ can be factored into linear terms and so $\sqrt{c_i} \in \mathbb{C}$ is a root. Then we can set $X_i' = \sqrt{c_i} X_i$ when $c_i \neq 0$ and $X_i' = X_i$ when $c_i$ is zero. It follows that any two conics with the same rank are projectively equivalent. Suppose $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are projectively equivalent. By Lemma 2, we have that we $f(x) = x^T M x$ and $g(x) = x^T B^T M B x$ where $B$ is invertible. Since $M$ and $B^T M B$ have the same rank, $\text{rank}(\mathbb{V}(f)) = \text{rank}(\mathbb{V}(g))$.

It follows immediately from Theorem 2 that all conics in $\mathbb{CP}^2$ of rank 3, that is all non-degenerate conics, are projectively equivalent. Note that the above theorem does not hold for fields that are not algebraically closed. In particular over $\mathbb{R}$, $C_1 = \mathbb{V}(X^2 + Y^2 + Z^2 = 0)$ and $C_2 = \mathbb{V}(X^2 + Y^2 - Z^2)$ have the same rank but are not projectively equivalent since $C_1$ is the empty set and $C_2$ is a circular cone.

The following example shows that a parabola is projectively equivalent to a unit circle.
Example 4. Consider the parabola in $\mathbb{V}(yz - x^2) \subset \mathbb{P}^2$. Note that we can write the defining equation of this variety as

$$f(x) = x^T M x = x^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix} x$$

and consider

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Note that $BMB^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$g(x) = x^T B^T MBx = X^2 - Y^2 + Z^2$$

Performing a change of coordinates $X' = X$, $Y' = Z$ and $Z' = Y$, we obtain $g(x) = X'^2 + Y'^2 - Z'^2$, which is a circle in $\mathbb{P}^2$.

4. Five Points Determine a Conic

In this section, we answer the questions presented in the introduction:

(1) How many points determine a conic?

(2) How many conics do we expect to pass through $m$ points and are tangent to $n$ lines?

To answer these questions, we need a very useful result that counts the number of intersections of varieties. Surprisingly, the same result also accounts for varieties that do not intersect in an affine patch as shown in an example below.

Lemma 3. Bezout’s Theorem Consider two varieties $V, W \subset \mathbb{P}^2$ defined by polynomials of degrees $m$ and $n$ respectively. If the varieties have no common component, then $\#(X \cap Y) = mn$. There are $mn$ distinct points if the varieties are not tangent to each other at any point (for proof, see [6]).

Example 5. Consider two circles $C_1 = \mathbb{V}(X^2 + Y^2 - Z^2)$ and $C_2 = \mathbb{V}(X^2 - 2XZ + Y^2)$. Since $C_1$ and $C_2$ are defined by degree 2 polynomials, $\#(C_1 \cap C_2) = 2 \cdot 2 = 4$ by Bezout’s Theorem. We immediately see that $[\frac{1}{2} : \frac{1}{2} : 1], [\frac{1}{2} : -\frac{3}{2} : 1] \in C_1 \cap C_2$. When $Z = 0$, the defining polynomials for $C_1$ and $C_2$ are both $X^2 + Y^2 = 0$. Then $Y = \pm iX$ and so $[1 : i : 0], [1 : -i : 0] \in C_1 \cap C_2$.

Example 6. Consider two circles $C_1 = \mathbb{V}(X^2 - 4XZ + 3Z^2 + Y^2)$ and $C_2 = \mathbb{V}(X^2 + 4XZ + 3Z^2 + Y^2)$. Note that in the affine patch, $Z = 1$, $C_1$ and $C_2$ do not intersect each other. However, by Bezout’s theorem, we expect that $\#(C_1 \cap C_2) = 4$. Subtracting the two defining polynomials, we obtain $8XZ = 0$. When $Z = 0$, $X$ is free and so $Y = \pm iX$. Then $[1 : i : 0], [1 : -i : 0] \in C_1 \cap C_2$. When $X = 0$, $Z$ is free and so $Y = \pm \sqrt{3}Z$. Then $[0 : \sqrt{3} : 1], [0 : -\sqrt{3} : 1] \in C_1 \cap C_2$.

We can extend Lemma 3 to projective spaces of higher dimension. We state the following without proof. For a proof, see [7].
Lemma 4. **Bezout’s Theorem for Higher Dimensions**

If \( V \subset \mathbb{P}^n \) is a complete intersection, i.e. an intersection of \( c \) hypersurfaces \( F_1, \cdots, F_c \) where \( c = \text{codim} V \), then

\[
\deg V = \deg F_1 \cdot \deg F_2 \cdots \deg F_c
\]

We now have the tools to answer the questions above.

To determine how many points determine a conic, recall that a conic on the projective plane is the variety

\[ C = \mathbb{V}(aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2) \subset \mathbb{P}^2 \]

This variety is determined by a non-zero \((a, b, c, d, e, f) \in \mathbb{C}^6\). Note that any \( \lambda(a, b, c, d, e, f) \) where \( \lambda \neq 0 \in \mathbb{C} \) defines the same conic. Therefore, we can identify a conic \( C \subset \mathbb{P}^2 \) with points \([a : b : c : d : e : f] \in \mathbb{P}^5\). Note that using this identification, conics passing through a fixed point \([\alpha : \beta : \gamma] \in \mathbb{P}^2\) form a hyperplane \( H_i \subset \mathbb{P}^5 \). Using this concept, we show that five points determine a conic.

**Theorem 3.** Given five points in \( \mathbb{P}^2 \), there exists a non-degenerate conic containing them. The conic is unique unless four of the points are collinear.

**Sketch of proof.** As discussed above, conics passing through a fixed point form a hyperplane \( H_i \subset \mathbb{P}^5 \). It follows that conics passing through five points form an intersection of five hyperplanes. Recall that the intersection of two unique hyperplanes \( H_1 \) and \( H_2 \) has dimension 4. Intersecting another hyperplane causes the dimension of the intersection to drop by one unless there is a dependence relation among the three hyperplanes. Using properties of determinants, it can be shown that four collinear points impose a dependence relation among the hyperplanes. Thus, if the hyperplanes are linearly independent, \( \dim(H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5) = 0 \). This implies that the intersection of the hyperplanes is a point in \( \mathbb{P}^5 \) and thus identifies a unique conic in \( \mathbb{P}^2 \).

It turns out that the conic is non-degenerate unless three of the points are collinear. An elegant proof using analytic geometry involves explicitly constructing an irreducible homogenous degree 2 polynomial in \( \mathbb{C}[X,Y,Z] \). We refer the reader to [8].

Since five points determine a conic, we only need to consider the case where \( m + n = 5 \) in answering the next question.

**Theorem 4.** There \( 2^n \) conics, some of which are degenerate, that pass through \( m \) points and are tangent to \( n \) lines, when \( m + n = 5 \).

**Proof.** [9] Consider a point \([\alpha, \beta, \gamma] \in \mathbb{P}^2\). From an earlier discussion, conics passing through that point form a hyperplane \( H_i \subset \mathbb{P}^5 \).

Now consider the line \( l = \mathbb{V}(AX + BY + CZ) \subset \mathbb{P}^2 \). Note that the points in this variety satisfy the equation

\[
X = -\frac{B}{A}Y - \frac{C}{A}Z
\]

Substituting for \( X \) in the defining equation for \( C \) gives

\[
(BY + CZ)^2 - bAY(BY + CZ) - cA^2Y^2 - d(BY + CZ)AZ + eA^2YZ + fA^2Z^2 = 0
\]
Note that this is a homogeneous degree 2 equation in $Y$ and $Z$:
\[ \epsilon_1 Y^2 + \epsilon_2 YZ + \epsilon_3 Z^2 = 0 \]
where $\epsilon_1 = cA^2 - bAB$, $\epsilon_2 = 2BC - bAC - dAB + eA^2$ and $\epsilon_3 = C^2 + fA^2$.

Recall that $l$ is tangent to $C$ when the discriminant $\epsilon_1 \epsilon_3 - \epsilon_2^2 = 0$. That is when
\[
(e^2 - 4cf)A^2 + (4bf - 2de)AB + (d^2 - 4af)B^2 + (4cd - 2be)AC \\
+(4ae - 2bd)BC + (b^2 - 4ac)C^2 = S_i = 0
\]

Fix the point $[A, B, C] \in \mathbb{P}^2$. Then the above equation is a variety $G_i = \mathbb{V}(S_i) \subset \mathbb{P}^5$. Since we associate a conic to a point in $\mathbb{P}^5$, conics that are tangent to $l$ are precisely the $\mathbb{V}(S_i)$ described above.

From the above discussion, we can easily find how many conics pass through 3 points and tangent to 2 lines. There are $\#(H_1 \cap H_2 \cap H_3 \cap G_1 \cap G_2) = 1^3 \cdot 2^2$ of them, by Lemma 4. Hence, there are $2^n$ conics passing through $m$ points and tangent to $n$ lines. \qed

**References**


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