Chapter 14

Counting Finite Sets

Somewhere in this chapter, Eccles says “...the reader may consider that the author has taken leave of his senses.” You may feel the same way about me, or yourself, as you try to understand the questions raised in this chapter. You may feel that the author is demanding an explanation of why one and one is two, or maybe he’s asking if the sky is really blue or just looks that way. Part of my job today is to convince you that serious questions are being asked, and that definitive answers are possible.

I think Chapter 10 in Eccles, which we are going to talk about today, is one of the best-written and most interesting chapters in the text.

We want to show first that we can count the number of elements in a finite set, and that every time we do so we will get the same result. In cognitive science this is called conservation of number, and it has to be rediscovered by every child. Perhaps mathematics could just assume conservation of number, but we are able to derive this principle from more basic concepts. Mathematicians always try to reduce their hypotheses to the minimum. Whenever two concepts are logically related, mathematicians try to make one an axiom and the other a theorem. Sometimes the most efficient set of basic axioms is not the most obvious, and more intuitive ideas may be derived from less obvious foundations. In our case, we will see that conservation of number is derived from the fact that the composition of bijective functions is bijective.

Your homework for Thursday deals with complicated counting problems, and I will show you some examples. However, I want you to see some of the foundational developments first. We need a theorem that we’ve already proved:

**Theorem 22** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be functions. If $f$ and $g$ are bijective, then $g \circ f$ is bijective.

**Proof.** You proved that the composition of surjective functions is surjective in the homework. The midterm included a proof that the composition of injective functions is injective. Thus the composition of bijective functions is bijective. ■

One of the problems mathematicians sometimes face when dealing with the foundations of their subject is how to define commonplace concepts. How, for example, can you define the number of elements in a set? Let’s begin with a key observation:

**Definition 23** $\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$.

**Proposition 24** Suppose we have a bijective function $f : \mathbb{N}_n \rightarrow \mathbb{N}_m$. Then $n = m$.

**Remark 25** This is the result that leads Eccles to suggest he has taken leave of his senses. How could anything be more obvious? And how could you prove it? Eccles puts the proof off for one chapter, but we’ll do it now.

**Proof.** The proof is by contradiction. Suppose $n \neq m$. Replacing $f$ by $f^{-1}$ if necessary, we may suppose $n > m$. Construct $g : \mathbb{N}_m \rightarrow \mathbb{N}_n$, $g(i) = i$. This is defined because $m < n$. The map $g$ is injective, but $g$ is not surjective because there is no element $x \in \mathbb{N}_m$, $g(x) = n$. Thus $g \circ f : \mathbb{N}_n \rightarrow \mathbb{N}_n$ is injective but not surjective. To get a contradiction and complete the proof, we will show by induction on $n$ that any injective map $h : \mathbb{N}_n \rightarrow \mathbb{N}_n$ must be surjective.
First the base case $n = 1$. If we have a map $h : \{1\} \rightarrow \{1\}$, then the only possible value for $h(1)$ is 1, so $h$ is surjective. (For this base case we did not have to assume that $h$ was injective because all maps $h : \{1\} \rightarrow \{1\}$ are injective.)

Now for the inductive step. Suppose we have an injective function $h : \mathbb{N}_n \rightarrow \mathbb{N}_n$. We want to show that $h$ is surjective. We may assume that any injective function $k : \mathbb{N}_{n-1} \rightarrow \mathbb{N}_{n-1}$ is surjective. We distinguish three possible cases

1. $h(n) = n$
2. There exists $j \in \mathbb{N}_{n-1}$ such that $h(j) = n$.
3. For all $j \in \mathbb{N}_n$, $h(j) < n$.

We will show, in case (1) and (2), that $h$ is surjective, and we will show that case (3) leads to a contradiction.

1. Since $h$ is injective, $i < n$ implies $h(i) < n$. Thus the restricted map $h : \mathbb{N}_{n-1} \rightarrow \mathbb{N}_{n-1}$ and $h$, restricted to $\mathbb{N}_{n-1}$, is still injective. By the inductive hypothesis (we have to use it somewhere), the restricted map $h : \mathbb{N}_{n-1} \rightarrow \mathbb{N}_{n-1}$ is surjective. Therefore $h : \mathbb{N}_n \rightarrow \mathbb{N}_n$ is surjective.

2. Define a function $k : \mathbb{N}_n \rightarrow \mathbb{N}_n$ by
   \[
   k(j) = h(n),
   k(n) = h(j) = n,
   k(i) = h(i) \text{ for } i \neq j \text{ and } i \neq n.
   \]

The function $k$ is $1 \mapsto 1$ and satisfies condition (1). Therefore $k$ is surjective, and thus $h$ is surjective.

3. The restricted map $h : \mathbb{N}_{n-1} \rightarrow \mathbb{N}_{n}$ is actually a map $h : \mathbb{N}_{n-1} \rightarrow \mathbb{N}_{n-1}$. Since the restricted map is injective, it is also surjective. But this contradicts the injectivity of $h$, because there exists $j < n$ such that $h(j) = h(n)$. Therefore case (3) leads to a contradiction.

\[
\]

**Definition 26** Suppose $A$ is a set. We say $|A| = n$ or the cardinality of $A$ is $n$ if there exists a bijective function $f : \mathbb{N}_n \rightarrow A$. (Later we will extend the notion of cardinality to values other than non-negative integers.) In this case, we say that $A$ is a finite set. Otherwise we say that $A$ is an infinite set.

We say $|\emptyset| = 0$.

**Proposition 27** The cardinality of a finite set $A$ is unique.

**Proof.** Suppose $f : \mathbb{N}_n \rightarrow A$ and $g : \mathbb{N}_m \rightarrow A$ are bijective. Then $g^{-1} \circ f : \mathbb{N}_n \rightarrow \mathbb{N}_m$ is bijective, so $n = m$. ■

**Proposition 28** Two sets $A$ and $B$ have the same cardinality if and only if there is a bijective function $A \rightarrow B$.

**Proof.** If $A$ and $B$ have the same cardinality $n$, then there exist bijective functions $f : \mathbb{N}_n \rightarrow A$ and $g : \mathbb{N}_n \rightarrow B$. Thus $g \circ f^{-1} : A \rightarrow B$ is bijective. Conversely, suppose $f : A \rightarrow B$ is bijective and $|A| = n$. Then there is a bijective function $g : \mathbb{N}_n \rightarrow A$. But using $g$ and $f$ we can construct a bijective function $f \circ g : \mathbb{N}_n \rightarrow B$, so $|B| = n$ as well. ■

That’s enough abstract nonsense. Let’s consider how to count things.

**Proposition 29** Let $A$ and $B$ be disjoint finite sets ($A \cap B = \emptyset$). Then $A \cup B$ is finite and $|A| + |B| = |A \cup B|$.

**Remark 30** Children learn that when you take two blocks, then three more blocks, you have five blocks.
We can write
\[ h(i) = \begin{cases} f(i) & \text{if } i \leq n \\ g(i-n) & \text{if } i > n \end{cases} \]

To show \( h \) is surjective, suppose \( x \in A \cup B \). If \( x \in A \) then \( x = f(i) = h(i) \) for some \( i, 1 \leq i \leq n \). If \( x \in B \) then \( x = g(i) = h(n+i) \) for some \( i, 1 \leq i \leq m \) or \( n+1 \leq n + i \leq n + m \). Thus for all \( x \in A \cup B \) there is \( i \in \mathbb{N}_{n+m} \) such that \( h(i) = x \).

To show \( h \) is injective, suppose we have integers \( i, j \) with \( 1 \leq i < j \leq n + m \). We must show \( h(i) \neq h(j) \).

If \( i \leq n \) and \( j > n \) then \( h(i) \in A \) and \( h(j) \in B \). Since \( A \cap B = \emptyset \) (we have to use this hypothesis somewhere), \( h(i) \neq h(j) \). If \( j \leq n \) then \( h(i) = f(i) \) and \( h(j) = f(j) \). Since \( f \) is injective, \( h(i) \neq h(j) \). Similarly, if \( i > n \) then \( h(i) = g(i-n) \) and \( h(j) = g(j-n) \). Since \( g \) is injective, \( h(i) \neq h(j) \).

**Remark 33**

In the next chapter, we will prove that any subset of a finite set is finite, so the only hypotheses needed for this corollary is that \( X \) is finite.

**Proof.**

A and \( X - A \) are disjoint, and \( X = A \cup (X - A) \).

**Corollary 32 (not in book)**

If \( A \subset X \) and \( X, A, X - A \) are all finite sets, then \( |X| = |A| + |X - A| \).

**Proposition 34**

Let \( A \) and \( B \) be finite sets. Then \( |A \times B| = |A| |B| \).

**Proof.** If either \( A \) or \( B \) is empty then \( A \times B \) is empty and the proposition just says \( 0 = 0 \) (which is true).

If \( A \) has a single element, \( A = \{a\} \) and \( |A| = 1 \). There is a bijection \( B \to A \times B \) given by \( b \to (a, b) \).

Thus \( |A \times B| = |B| = |A| |B| \).

In general, \( A \times B \) is the disjoint union of the sets \( C_a = \{a\} \times B \) for all \( a \in A \). Thus by the corollary

\[ |A \times B| = \sum_{a \in A} |C_a| = \sum_{a \in A} |B| = |A| |B| \]

The next proposition is the most important of all for counting.

**Proposition 35**

If \( A \) and \( B \) are finite sets then \( |A \cup B| = |A| + |B| - |A \cap B| \).

**Proof.** We can write \( A \) as a disjoint union:

\[ A \cup B = (A - B) \cup (B - A) \cup (A \cap B) \]

Since \( A \) is the disjoint union \( A = (A - B) \cup (A \cap B) \) and \( B \) is the disjoint union \( B = (B - A) \cup (B \cap A) \), we have \( |A| = |A - B| + |A \cap B| \) and \( |B| = |B - A| + |B \cap A| \). Thus

\[ |A \cup B| = |A - B| + |B - A| + |A \cap B| = |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| = |A| + |B| - |A \cap B| \]
Corollary 36 If $A, B, C$ are disjoint finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |C \cap A| - |A \cap B| + |A \cap B \cap C|$$

**Proof.** We apply the proposition three times:

$$|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$$
$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$
$$= |A| + |B| - |A \cap B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Here’s an example (from the text). Suppose you have 144 tiles that are either triangular or square. Each tile is either red or blue, wooden or plastic. There are 68 wooden tiles, 69 red tiles, 75 triangular tiles, 36 red wooden tiles, 40 triangular wooden tiles, 38 red triangular tiles and 23 red wooden triangular tiles. How many blue plastic square tiles are there?

Let $L$ be the set of tiles, $T$ the triangular tiles, $R$ the red tiles and $W$ the wooden tiles. Then, for example, $T \cap R$ is the collection of red triangular tiles and $L - R$ is the collection of blue tiles. We know:

$$|L| = 144$$
$$|T| = 75$$
$$|R| = 69$$
$$|W| = 68$$
$$|R \cap W| = 36$$
$$|T \cap W| = 40$$
$$|T \cap R| = 38$$
$$|T \cap W \cap R| = 23$$

Let $P = W^c = L - W$ be the plastic tiles, $B = R^c = L - R$ be the blue tiles and $S = T^c + L - T$ be the square tiles. Then

$$P \cap B \cap S = W^c \cap R^c \cap T^c$$
$$= (W \cup R \cup T)^c$$
$$= L - (W \cup R \cup T)$$

Thus our desired answer is:

$$|P \cap B \cap S| = |L| - |W \cup R \cup T|$$
$$= |L| - (|W| + |R| + |T| - |R \cap T| - |T \cap W| - |W \cap R| + |W \cup R \cup T|)$$
$$= 144 - (68 + 69 + 75 - 38 - 40 - 36 + 23)$$
$$= 23$$