Riemann Hypothesis

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12\textsuperscript{th} April 2011
Abstract

Riemann Hypothesis is one of the most important unsolved math problems in the world. It is a conjecture about the zeros of the Riemann zeta function. Although it has a very simple general form, mathematicians tried to completely solve it generation by generation. This paper is actually an interesting story that can take you get comprehension into the question and introduce how those mathematical masters get the solution step by step. At last, there are some introductions to the extended Riemann Hypothesis.

In this paper, I’ll mainly talk about Riemann Hypothesis. Now, let me started with a question. It is widely known for most mathematical students that the mathematical constant e, which was firstly used by Swiss mathematician and physicist Leonhard Euler, is an expression of

\[ e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \]

but it also can be written as

\[ \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \]

Sometime, when we add a list of convergent numbers together, we can get a very “good-looking” constant. So why don’t we have a new audacity of imagination that
what happens if the following infinite numbers sums up:

\[ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \ldots \]

(for most math major students, we know this list of numbers is not convergent, so it’s hard to get the answer here)

Or what happens if we sum another list of numbers:

\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \ldots \]

If anybody solved this problem by yourself and think that you are the first one to find this answer, I feel so sorry to tell you that Euler solved them over two hundred years ago. It is very interesting that he found

\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \ldots = \frac{\pi^2}{6} \]

And

\[ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \ldots = \frac{\pi^4}{90} \]

Now you must begin to think about \( \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots \) or even \( \frac{1}{1^{20}} + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \frac{1}{4^{20}} + \ldots \)

How about let’s start to study a totally new expression:

\[ Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \]

This function is actually called Riemann zeta function and the emphasis of this paper will be a conjecture about the distribution of zeros of this function, which is the
famous Riemann hypothesis. Now, let me introduce some background information and development of it.

First, I need to mention several important researchers for Riemann hypothesis.

Riemann Bernard is a famous German mathematician who had great contribution to analysis and differential geometry, some of them enabling the later development of general relativity. He introduced the Riemann zeta function and established its importance for understanding the distribution of prime numbers in a single short paper.

David Hilbert is another famous German mathematician in history. He discovered and developed a broad range of fundamental ideas in many areas, including invariant theory and the axiomatization of geometry. He is also the founder of the famous Hilbert spaces which is one of the foundations of functional analysis. The 23 unsolved problem he posted at the international Congress of Mathematicians in Paris in 1900 is very influential, one of the problem is the Riemann Hypothesis.

Barry Charles Mazur is a professor of mathematics at Harvard. He proved the generalized Schoenflies conjecture, Mazur manifold and Mazur swindle. He observed in 1960s on the analogies between primes and knots.

Don Bernard Zagier is an American mathematician whose main work is number theory. His famous work is a joint work with Benedict Gross. He is also famous for discovering a short and elementary proof of fermat’s theorem on sums of two squares. His paper “The First 50 Million Prime Numbers”, which is related to Riemann Hypothesis, is also famous.
Now, let’s look at some history of Riemann Hypothesis

Riemann Hypothesis was firstly proposed in 1859 by Riemann Bernard. It is a very funny that he only wrote one paper about number theory in his life, but this paper already became one of the most important papers in number theory and we have to admit that most of the number theorists who worked for number theory for whole life can’t bring one paper as important as this paper in which, as stated above, introduced the Riemann zeta function and established its importance for understanding the distribution of prime numbers.

At very beginning of 20th century, there was a German mathematician called David Hilber also proposed 23 unsolved problems that he thought would be very important in 20th century, Riemann Hypothesis is one of these problems. Until today, seventeen of these problems are solved or partially solved. “For many of these questions, the solutions (or partial solutions) resulted in breakthroughs that became fundamental to the development of 20th century mathematics.” (Thomas) However, Riemann hypothesis is not one of the solved problems.

Exactly one hundred years later, the Clay Mathematics Institute in Cambridge, Massachusetts also proposed 17 unsolved problems in 2000 and Riemann Hypothesis appeared on the list again. This time, they offered one million US dollar reward to each single problem’s solution. Although the great interests attract a lot of mathematicians to solve them, there are still many problems remain unsolved. “One such problem, known as the Riemann Hypothesis, is particularly interesting because of the simplicity with which it can be stated, the incredible difficulties encountered
when trying to attack it, and the expected utility of a solution.” (Harrison)

Now we return to the function $Z(s)$ and study its characteristics:

We already put some numbers bigger than 0 into $Z(s)$, now let’s try to put 0 or -1 into $Z(s)$:

$$Z(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = 1 + 1 + 1 + ... = 1 + 1 + 1 + ...$$

$$Z(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + 1 + 1 + ... = 1 + 2 + 3 + 4 + ...$$

We can easily find that both of the summations above will go to infinity and we already know that $Z(1)$ doesn’t has a “good-looking” result, besides, we can boldly guess that $Z(s)$ will goes to infinity as well when $s<-1$. So at this point, we can draw an interesting conclusion that in order to let $Z(s)$ converge, $s$ should be bigger than 1!

When we study a function, a good idea is that we can change it to another form and see what happens. Here, I should admit that Leonhard Euler is absolute genius because he is not only a pioneer in many scientific areas such as infinitesimal calculus and graph theory, but also a “freak” who found many significant details in number theory. Euler showed to us that the Riemann zeta function can actually be written as the product over the set of primes:

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...$$

$$= (\frac{1}{1-\frac{1}{2^s}})(\frac{1}{1-\frac{1}{3^s}})(\frac{1}{1-\frac{1}{5^s}})(\frac{1}{1-\frac{1}{7^s}})...$$
By this expression, we can change this function from a statement about integers into a statement about prime number. Jason says “the above representation of zeta-function is our first suggestion that the Riemann Hypothesis might have consequences in number theory, and indeed there is a more purely number theoretic form of the hypothesis.” (Jason, 2003, *The extended Riemann hypothesis and its application to computation*)

Now, it’s the time that we continue our exploration of finding zeros of the Riemann zeta function. We already know that $Z(s)$ doesn’t converge when $s \leq 1$, the Riemann hypothesis talks about the zeros outside the region of convergence of this series, in other words, we should only think about the solutions when $s<1$. So we must have an analytic continuation of this Riemann zeta function that its radius of convergence goes beyond the domain of its original function($\text{Re}(s)>1$) and reaches to the whole complex plane. Riemann came up with the form of this continuation:

$$
\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}
$$

(p is the prime number)

It can also be expressed as:

$$
\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)
$$
We can find that \( \zeta(2) = \frac{\pi^2}{6} \), \( \zeta(4) = \frac{\pi^4}{90} \) and so on, that means \( \zeta(s) = Z(s) \) for \( s > 1 \) and we can notice that \( \zeta(1) \) is undefined because \( s = 1 \) is the pole.

By the second expression above, we can find that when \( s \) equals negative integer, the factor \( \sin\left(\frac{\pi s}{2}\right) \) becomes zero, and the entire right side of the function becomes zero.

So, \( \zeta(s) = 0 \) when \( z = -2, -4, -6... \), and these answers are called trivial zeros of the generalized zeta function.

Besides the negative even integers, there are also many zeros lies in the region \( 0 < \sigma < 1 \) which is called the critical region and these zeros are called non-trivial zeros.

Danish mathematician J.P. Gram in 1903 firstly found that non-trivial zeros lies on the line \( s = \frac{1}{2} + it \), for example:

\[
\zeta\left(\frac{1}{2} + 14.134725142i\right) = 0
\]

and \( \zeta\left(\frac{1}{2} + 21.02039639i\right) = 0 \)

(In fact, there are many, actually millions of them, \( r \) in \( s = \frac{1}{2} + it \) satisfying \( \zeta(s) = 0 \))

At this point, we can have a strong conclusion that \( \zeta(s) = 0 \) when \( s \) is a negative even integer or \( s = \frac{1}{2} + it \) for some values of \( t \) (be careful, not all \( t \) in \( s = \frac{1}{2} + it \) satisfying \( \zeta(s) = 0 \) , the finding of “i” still keep going in the mathematician filed, the tables known “i” is shown in the *Tables of the Riemann zeta function written by Haselgrove in 1960*) Unfortunately, that doesn’t mean we already know how to solve the Riemann Hypothesis, which is the conjecture about the distribution of zeros of the Riemann zeta function. Actually it is still not totally solved even today although it has been
proposed over 150 years. Therefore, most mathematicians think it is the most important unsolved problem on the earth!

Now we come to the application of Riemann Hypothesis, It is very easy for all of us to count how many primes under 10 or 20, because we can just write down 1,2,3,5...and add them together. But have any of you thought about how many primes under 10000 or even 100000. We can’t just simply count this time and I know this question must make you crazy. However, another master named Carl Friedrich Gauss, who is a great and famous German mathematician, helped us to solve these problems about 160 years ago. “Gauss, in a letter to the astronomer Hencke in 1849, stated that he had found in his early years that the number $\pi(x)$ of primes up to $x$ is well approximated by the function

$$Li(x) = \int_2^x \frac{dt}{\ln t}$$

(This function is called logarithmic integral)

Li (x) gave us an advance about understanding the prime numbers. But we should notice that Li(x) is just an estimate of the real number of primes under $x$, which is denoted by $\pi(x)$ in math, that means there is a certain level of error existing between Li(x) and $\pi(x)$. In fact, “Li(x) and $\pi(x)$ can differ as much as $\sqrt{x} \cdot \ln x.$” (Thomas).

“In the last hundred years, mathematicians have come up with better and better answers, but we’re still nowhere near where the data indicates that we should be.”

On the other hand, we can have a conclusion that if the Riemann Hypothesis is true, then Li(x) and $\pi(x)$ can differ no more than $\sqrt{x} \cdot \ln x$. So at this point, we have the
function:

\[ \pi(x) = li(x) + O(\sqrt{x} \ln(x)) \quad \text{or} \quad \pi(x) = li(x) + O(x^{1/2+\varepsilon}) \]

Now let me introduce some extended Riemann Hypothesis:

(1) \( L_p(s) \) is a function of a complex variable \( s \):

\[ L_p(s) = \sum_{n=1}^{\infty} \left( \frac{n}{p} \right) \frac{1}{n^s} \]

Where \( \left( \frac{n}{p} \right) \) is the Legendre Symbol, define for \( p \) prime as

\[ \left( \frac{n}{p} \right) = 0 \text{ if } p \text{ divides } n \]
\[ = 1 \quad \text{if } n \text{ is a quadratic residue mod } p \]
\[ = -1 \quad \text{if } n \text{ is quadratic nonresidue mod } p \]

All the zeros of \( L_p(s) \) which lies in the “critical strip” have real part equal to 1/2.

(2) For \( n \) and a relatively prime integers and \( \varepsilon > 0 \)

\[ \pi(x,n,a) = \frac{Li(x)}{\phi(n)} + O(x^{1/2+\varepsilon}) \]

\( \pi(x,n,a) \) is the number of primes less than or equal to \( x \) and equivalent to \( a \) mod \( n \).

Riemann Hypothesis is still a famous unsolved problem, so let’s look at some current results about it.

In the article “What is Riemann Hypothesis” written by Barry Mazur and his partner William Stein in 2007, he reformulated the Riemann Hypothesis and seek the information in the staircase of primes, and introduce the spike distribution. in the
The first 50 Million Prime Numbers" (1977), Don Zagier shared his thoughts about finding the distribution of prime numbers in a contemporary mathematician view by using various methods.

Bibliography


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