Abstract. The Kepler’s sphere packing problem was a problem questioned by a famous astronomer named Johannes Kepler in 1611. The problem asks what is the most efficient way of packing the most amounts of equal-sized spheres into a large square crate, and eventually in a cube of infinite volume. There are two suggested ways of packing the spheres: one way is to perfectly pack the spheres on top and to the sides of each other, and the other way is to pack the spheres side to side, then the top ones will be sitting on between the two spheres on the bottom. The conjecture was not proven until recently by Dr. Thomas Hales, who was caught in a controversy for the validity of his method to the proof. This fascinating problem can be very helpful to many grocery stores since they use it to pack their round fruits in the most efficient way. Also, it will contribute to information technology by compacting data into the most efficient space possible.

1. Introduction

1.1. Johannes Kepler. Johannes Kepler was born on December 27, 1571 in Weil der Stadt, Germany. He was an astronomer, famous for his three laws of planetary motion and many other astronomical works [Vo99 10,65,92]. Though he was a famous astronomer, Kepler was also a famous mathematician of his time. His famous Kepler’s conjecture began when Sir Walter Raleigh requested a formula to calculate the number of cannonballs he had piled upon the deck of his ship. Thomas Hariot gave Sir Rayleigh the formula, but he could not figure out the most efficient way to stack the cannonballs, so he passed the problem to Kepler [Dev]. Kepler answered the problem without any justification by stating that the Creator’s will was to have twelve seeds arranged around a central seed for the tightest packing [Sz03 20-22]. Though he was correct, no mathematicians were able to prove his claim even after his death in November 15, 1630, making his sphere packing problem a conjecture. His famous conjecture had passed on to this day and countless mathematical contributions were made thanks to Johannes Kepler’s clever statement.

1.2. Kepler’s Conjecture. The Kepler’s conjecture’s main idea is to find the densest possible pack of equal radii spheres in a cube of infinite volume. The answer to the problem was already given by Kepler and all mathematician agreed with it. However, the real problem lies with an official proof for Kepler’s answer, and for hundreds of years, there were none. Finally in August 8, 1998, Dr. Thomas Hales claimed to have proven the conjecture [Sz03 201]. However, controversies arose for his proof among the math communities and the status of his proof was debated.
2. The Early Contributions of Newton, Gregory, and the Twelve Spheres

Isaac Newton and David Gregory, two famous mathematicians, had a history of debating over how many equal radii spheres can be arranged to touch a given sphere. Their problem sounded just like the Thomas Hariot’s inquiry, but instead of cannonballs, they debated over general spheres in the third dimensions. They both agreed that in two-space dimensions, no more than six pennies can touch a given penny [Ha06], also called a hexagonal packing [Sz03 3-4]. Despite their previous agreement, in three-space dimensions, their opinions diverged. Newton argued that a maximum of twelve spheres could touch a given sphere, while Gregory argued for thirteen spheres instead. Newton had the correct answer, but no official proof was given until 1953 by Schtte and van der Waerden [Ha06]. As it turned out, Newton’s packing were called hexagonal close packing (hcp) and/or face-centered cubic packing (fcc).

3. The Densest Types of Sphere Packing

3.1. Hexagonal Close Packing. The hexagonal close packing (hcp) is a type of stacking and packing of spheres together in the third dimensions (possibly in higher dimensions, as well). It was discovered by mathematicians whom realized that dimples were formed between every three melons of the first layer of melons. In order to maximize the density, the melons were to be placed on top of the dimples for the second layer above. Then, the steps were repeated for each layers of melons. From their observations and experiments, the hcp was found and it turned out to be the densest packing of spheres with a density of 74.05 percent [Sz03 7].

Another definition for hcp was made by Dr. Thomas Hales in one of his official papers. He said, “in the hcp, each ball is tangent to twelve others. For each ball in the packing, the arrangement of twelve tangent balls is again the same. We call it the hcp pattern” [Ha06]. So far, the definition is no more different than the previous one. However, Dr. Hales added more details to the hcp. He continued, “in the hcp pattern, there is a plane through the center of the central ball that contain the centers of six other balls at the vertices of a regular hexagon” [Ha06]. Basically, the two hcp meanings are the same, as they both produce the same type of packing, but they both should be credited for their contribution to the meaning of the hcp.

3.2. Face-centered Cubic Packing. The face-centered cubic packing (fcc) is another type of sphere packing method in the third dimension. The method came to be in the old days when fruit vendors would pack their melons by having their first layer all lined up neatly, row-wise and column-wise. Then, the second layer of melons were to be placed on the dimples between four spheres, and the process was repeated for all the other layers of melons. Mathematicians previously thought that the hcp was the densest packing method, but it turned out that the fcc had a density of 74.05 percent, as well. Soon, mathematicians figured out that the hcp and fcp were the same except they were viewed in different points-of-view. In fact, there are infinite amounts of packing methods with a density of 74.05 percent [Sz03 8-9]. They are all the same answer to the Kepler’s conjecture that mathematicians were trying to prove.
Dr. Hales came up with his own definition for the fcc as well. He claimed that like the hcp, the fcc were the same, but the fcc pattern contained four different plane through “the center of the central ball that contain the centers of six other balls at the vertices of a regular hexagon” [Ha06], unlike the hcp pattern of one plane. The fcc was actually the main method of packing used to be proven by Dr. Hales for the Kepler’s conjecture, as well as many other mathematicians.

4. $\pi/\sqrt{18}$

The densest possible packing of spheres is the hcp, the fcc, and the other infinitely many ways to get 74.05 density of sphere packing. The importance of $\pi/\sqrt{18}$ is that there cannot be any other sphere packings in the third dimension with density more than that. In order to prove Kepler’s conjecture, $\pi/\sqrt{18}$ has to be proven to be true or by contradiction. Of course the proof is not that simple, but the density is very significant part of it. Currently, two modern mathematicians derived $\pi/\sqrt{18}$ in their own ways, and they are Dr. Wu-Yi Hsiang and Dr. Thomas Hales.

4.1. Hsiang’s $\pi/\sqrt{18}$. Dr. Wu-Yi Hsiang created his own conditions for his fcc packing, where it consisted of spheres of radii $1/\sqrt{2}$ centered at the fcc lattice points. Each cube of edge length 2 and vertices of even coordinates contained eight octant spheres and six half spheres and hence a total of four whole spheres [Hsi93]. He got the density of the fcc as

$$4 \cdot \frac{4\pi}{3} \left(\frac{1}{\sqrt{2}}\right)^3 / 8 = \pi/\sqrt{18} = 0.74048...$$

Dr. Hsiang came out with the density of the fcc by setting his own conditions and using geometry to find the density in his Kepler’s conjecture proof.

4.2. Hales’ $\pi/\sqrt{18}$. Similarily to Dr. Hsiang, Dr. Thomas Hales set his own conditions to get the fcc packing density. He let $S$ be a regular tetrahedron of side length 2. He got the fraction of the tetrahedral solid occupied by the part of the four balls within the tetrahedron as $\delta_{tet} \approx 0.7797$. He did something similar to an octahedron, and he got $\delta_{oct} \approx 0.72$. He concluded that density $\pi/\sqrt{18}$ of the fcc packing is a weighted average of $\delta_{tet}$ and $\delta_{oct}$ [Ha06]:

$$\pi/\sqrt{18} = \frac{1}{3} \delta_{tet} + \frac{2}{3} \delta_{oct}.$$  

Although, Dr. Hsiang and Dr. Hales came up with their own ways to find $\pi/\sqrt{18}$, it is fair to say that the conclusion of the densest possible packing in the third dimension has a density of 74.05 percent.

5. Wu-Yi Hsiang’s and the Kepler’s Conjecture

In 1993, Dr. Hsiang published his proof of the Kepler’s conjecture. In his proof, he defined local cells and their local density. He gave a theorem of optimal local density of sphere packing was equal to 0.75469.... He also gave a lemma of the global density of a packing was the weighted average of the locally averaged densities. Then, his second theorem claimed that $\pi/\sqrt{18}$ was the optimal upperbound of the locally averaged density of sphere packing [Hsi03]. Dr. Hsiang was sure his proofs of the theorems and lemma were correct, but it did receive much recognition by the mathematic communities. Many mathematicians found flaws in his paper as they went farther into his paper, and one of his biggest critics was Dr. Hales. Dr.
Hales claimed that Dr. Hsiang’s proof was based on too many inequalities and critical case analysis. The critical case analysis were experiments and examples Dr. Hsiang chose and tested. Dr. Hales said, “We should not underestimate the value of examples especially well-chosen ones, in providing heuristic evidence for difficult conjectures” [Ha94]. Dr. Hsiang used examples and experiments to prove parts of his proof, but Dr. Hales argued that his paper did not give a proof for the general Kepler’s problem. Unfortunately, in 1999, Dr. Hsiang was invited to the Institute of Advanced Studies’ workshop for a Kepler’s conjecture lecture, but the audiences were not convinced by his proof and speech [Sz03 204-205]. Yet again, the Kepler’s sphere packing problem remained as a conjecture.

6. Thomas Hales and the Kepler’s Conjecture

6.1. Short Biography. Thomas Hales is currently a Mellon Professor of Mathematics at the University of Pittsburgh. He went to Stanford University where he received his B.S. and M.S. degrees. He received a Tripos Part III from Cambridge University, and a Ph.D in representation theory at Princeton University. He was a faculty of the University of Michigan, where he first published his proof of the Kepler’s conjecture [HaBio].

6.2. Proof of Kepler’s Conjecture. In 1988, Dr. Hales and his graduate Sam Ferguson, proved Kepler’s conjecture. In August 9, 1998, Dr. Hales sent an e-mail to his colleagues around the world of his success [204-205]. His proof was mainly proven by computers, because he claimed the math was too hard to be done by hand. So he used his computer to calculate subjects such as inequalities by interval arithmetic, combinatorics, linear programming bounds, branch and bound methods, numerical optimization, and organization of output [Ha06]. However, Hales’ proof came into a controversy because of his intensive use of computer programs to prove the problem.

6.3. Controversy. In 1999, the Institutes of Advanced Studies held a workshop for the Kepler’s conjecture, where many mathematicians were invited including Dr. Hsiang and Dr. Hales. Along with Dr. Hsiang, Dr. Hales was given a chance to lecture and convince his proof to the audience, but unlike Dr. Hsiang, he was much more well received [Sz03 204-205]. However, he was not able to escape the controversy. It was stated in general, there are three types of proofs. The first grade proof, also the highest quality, is one that incorporates why and how the result is true. The second grade proof is a proof by contradiction, where the answer is set false and if it still turns out to be not false, then by contradiction it is true. The last type of proof, also the worse, is one with a large number of cases that are considered separately and are required to be verified one by one. Usually mathematicians use computers to prove the third grade types of proofs [Ca01 127]. Unfortunately for Dr. Hales, he used the worse case of proving the Kepler’s conjecture, so his method was not completely accepted. Although most mathematicians accept his work, there are still people like Dr. John H. Conway, a math professor at Princeton, who does not feel right with computer proofs [ch04].

7. Applications of Kepler’s Sphere Packing Problem

The conjecture is proven that the fcc is the best packing method of spheres in the third dimension. Undoubtedly, grocery vendors can learn to pack their fruits in
the most optimal way by using the fcc or hcp. Also, indirectly, the conjecture has contributed to the coding of error-detecting and error-correcting programs, where information are stored. The important part of the conjecture to information technology today is the math that mathematicians contributed for hundreds of years to the problem, helping programmers to code better packed informations [Dev]. Kepler would have never thought his problem of packing seeds would be used in digital bits hundreds of years later. Another problem relating to the conjecture is the Voronoi cells and the math to calculate the packing of atoms and its anatomies. It turns out by using Voronoi cells, mathematicians can help describe and analyze the electron density distribution[GHBB]. Mathematicians are finding more and more ways to use the Kepler’s conjecture for modern usage and technological advancements, which shows the significance of the problem itself.

8. Other Related Problems

The cannonball stacking of Sir Rayleigh and Newton-Gregory’s twelve spheres were both similar problems to the Kepler’s conjecture, and both of them were solved as previously stated. Also, the cannonball stacking was what led to the conjecture itself.

8.1. The Dodecahedral Conjecture. The Dodecahedral conjecture was first stated by Fejes-Toth. The conjecture was came upon because of Kepler’s conjecture, when Toth wanted to prove the problem. Toth believed that nature had an inclination to efficiency, like minimizing a volume or maximizing density. He believed regularity in nature is usually brought by some efficiency principles. He came up with a two-stage proof to prove Kepler’s conjecture. The first stage was to divide the space into Voronoi cells, which are polyhedra with walls running exactly halfway between two neighboring spheres. The second stage would consist of the search for cells with the smallest volume, and the dodecahedron would be a great candidate. Toth soon realized that V-cells cannot tile space, so the neighboring V-cells had to be of different shapes to fill in the gaps. However, when packing different V-cell shapes together, the cells are usually surrounded by large, loosely fitting V-cells, which meant the advantage of having a tight fit was ruined. Soon, he realized he was on the wrong path [Sz03 156-157]. The Dodecahedral conjecture was given in an article by Thomas Hales and Sean McLaughlin, and they followed the introduction with a proof [HaMcL]. Similar to Kepler’s conjecture, the Dodecahedral conjecture is the packing of Voronoi cells. As of today, it seems to have been proven by Hales and McLaughlin with Voronoi deformation, linear algebra, computers and more. Again, Hales used computers to help prove the conjecture, which may lead to more controversies. The future of mathematical proofs could be all proven by computer at this rate.

9. Conclusion

Since the unproven claim by Johannes Kepler, his sphere packing problem became a conjecture for hundred of years. Many mathematicians contributed to the slowly proven problem, such as Newton-Gregory’s debate, the discovery of the hexagonal close packing and the face-centered cubic packing with $\pi/\sqrt{18}$ as the densest packing, Wu-Yi Hsiang, Thomas Hales, and many other mathematicians. Eventhough, proofs had been coming and disappearing, finally, Dr. Hales had
proven the conjecture. Although his method of the proof is still not completely accepted, he is still recognized as the first person who proved the Kepler’s sphere packing problem by majority of mathematicians. Dr. Hales and Sean McLaughlin continued on to solve the Dodecahedral conjecture with computers, as well. Indirectly, the conjecture has led our information coding field to become more efficient, as well as an advancement in the physics of atoms. Undoubtedly, more advancements in result of the Kepler’s conjecture will surface and it is all thanks to the rigorous and intellectual of hundreds of years of contributions from mathematicians from all over the world. This is just the beginning of the real potential of the Kepler’s sphere packing problem since the future still has much room for improvement.

References

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