3x + 1

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Abstract

Take any arbitrary natural number $x$, apply the function: if $x$ is even, divide by two; if $x$ is odd, triple $x$ and add one. If we iterate this process, the $3x+1$ conjecture says that for any arbitrary $x$, the iterated function will terminate with the value 1. Although this seemingly trivial function can be easily understood by the grade school student, the proof of the conjecture has eluded all mathematicians. In this paper, I describe the history of the conjecture, outline some of what is known of the problem, and conclude with some personal thoughts.
1 Introduction

Let us begin with an example. Take an arbitrary “seed” number. If it is even, we halve it; if it is odd, we triple it and add one. Then we repeat this process and see where it goes (if anywhere).

Let’s take 27. It’s odd, so we triple it and add one and get 82. 82 is even, so we halve it and get 41. Continuing,

\[ 41 \rightarrow 124 \rightarrow 62 \rightarrow 31 \rightarrow 94 \rightarrow 47 \rightarrow 142 \rightarrow 71 \rightarrow 214 \rightarrow 107 \rightarrow 322 \rightarrow 161 \rightarrow 484 \rightarrow 242 \rightarrow 121 \rightarrow 364 \rightarrow 182 \rightarrow 91 \rightarrow 274 \rightarrow 137 \rightarrow 412 \rightarrow 206 \rightarrow 103 \rightarrow 310 \rightarrow 1 \]

In many ways, the “path” of this example mirrors that of the history of the 3x+1 conjecture and the futile attempts at proving where, if anywhere, the “path” takes us. Although this seemingly trivial function can be easily understood by the grade school student, the proof of the conjecture has eluded all mathematicians. In this paper, I describe the history of the conjecture, outline some of what is known of the problem, and conclude with some personal thoughts.
2 The Problem

Consider a mapping \( T : \mathbb{N} \to \mathbb{N} \) where

\[
T(x) = \begin{cases} 
\frac{x}{2} & \text{if } x \text{ is even}, \\
3x + 1 & \text{if } x \text{ is odd}.
\end{cases}
\]

Define the iterate of \( T \) as \([4]\):

\[
\begin{cases}
T^{(0)}(x) = x \\
T^{(i+1)}(x) = T(T^i(x))
\end{cases}
\]

Let us call the sequence of \( T^{(i)}(x) \) the trajectory of \( x \).

**The 3x+1 Conjecture:** For every \( x \in \mathbb{N} \), there is a finite \( k \) such that

\[
T^{(k)}(x) = 1.
\]

As Lagarias shows in his survey article, we can look at the 3x+1 problem through the lens of graph theory [3]. In graph theory, a graph consists of finite sets of objects called vertices and edges. A directed graph or digraph is a pairwise relation between a set of vertices and a set of edges [2]. For the 3x+1 problem, we can view the positive integers as the vertices and the mapping from \( x \) to \( T(x) \) as the edges. For our introductory example, our graph might look roughly like this: \( 27 \to 82 \to 41 \to \ldots \to 16 \to 8 \to 4 \to 2 \to 1 \). For \( x=336 \), the graph looks like: \( 336 \to 168 \to \ldots \to 32 \to 16 \to 8 \to 4 \to 2 \to 1 \). For \( x=133 \), the graph looks like: \( 133 \to 400 \to \ldots \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1 \). For \( x=341 \), the graph looks like: \( 341 \to \ldots \to 512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1 \). Clearly, the trajectory of \( x \) is unpredictably sporadic. However, looking at our small universe of examples, it seems that for
the trajectory of \( x \) to terminate at 1, it must merge at some point onto the “highway of \( 2^n \)” As Lagarias shows, there are three possible behaviors for trajectories when \( x > 0 \).

(i) Convergent trajectory. It converges to 1 (the so-called Collatz Conjecture).

(ii) A non-trivial loop. The sequence is cyclic (not at 1).

(iii) Divergent trajectory. The sequence goes to infinity [3].

None of the three possible trajectories have yet to be proven.

3 The History of the 3x+1 Conjecture

The conjecture is often attributed to Lothar Collatz, hence the term Collatz Conjecture. In 1932, while a student at the University of Hamburg, Collatz explored the function

\[
g(n) = \begin{cases} 
\frac{2}{3}n, & \text{if } n \equiv 0 \pmod{3}, \\
\frac{4}{3}n - \frac{1}{3}, & \text{if } n \equiv 1 \pmod{3}, \\
\frac{4}{3}n + \frac{1}{3}, & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

Specifically, his original question was whether \( g^{(k)}(8) \) was bounded or not. And it was Collatz’s interest in graph theory that led him to model his iteration problem using the structure of graphs [3]. Even though this original problem differs from our 3x+1 conjecture, it raises many of the same issues about the trajectory as our 3x+1 conjecture.

The problem circulated informally within the mathematics community for many years and as such took on different names like “Ulam’s problem,” after the physicist from Los Alamos, or “Kakutani’s problem,” a mathematician from Yale, or the “Syracuse problem,” after a colleague of Collatz visited Syracuse University [3].
4 What is Known

Stopping Time

One result by Terras addresses the stopping time problem on the positive integers [4]. To understand the idea of stopping time, let us note that the trajectory of $x$ ($x>1$), for it to terminate at 1, must, at some point, be less than the initial value $x$. Terras defined a function $\chi(x) = k$, called the stopping time of $x$, where $k$ is the smallest positive integer such that $T^{(k)}(x) < x$. And if no such integer exists, then, $\chi(x) = \infty$. He noted that since $\chi(0) = \chi(1) = \infty$, the 3x+1 conjecture can be restated as: $\chi(x)$ is finite for all $x \geq 2$. Terras, in his paper, demonstrated that $\chi$ has a well defined distribution function

$$F(k) = \lim_{x \to \infty} \left( \frac{1}{x} \right) \mu \{ n : n \leq x \text{ and } \chi(n) \geq k \}$$

where $\mu$ is the counting function. And here is the remarkable result: the limit of $F(k)$ exists and $F(k) = 0$ as $k \to \infty$. According to Lagarias, this means that, “almost all integers have a finite stopping time.” The implication here is that the 3x+1 conjecture holds for “almost all” large values of $x$. However, in the realm of mathematical proofs, “almost all” doesn’t quite have the definitiveness of other techniques, say induction.

A Probabilistic Argument

Crandall’s research paper of 1978 gives a probabilistic argument supporting the conjecture [1]. Here is my watered down version of the argument: If our initial value is even ($2k$), then the next term will be odd ($k$). Not if $k$ is even. Can you be a little more precise? Thus, we begin with an odd value ($2k+1$). The next term can either be odd or even, so we assume that the probability is
½ for both cases. We repeat this process what process? until we reach the next odd integer, and we assume the probability of what? to be ¼. And as we repeat this procedure, the growth factor between two consecutive odd integers is 3/4<1 [4]. The implication here is that the terms of the trajectory, on average, are “shrinking” thus a divergent trajectory is not possible [1]. Clearly the weakness of this argument is in the assumption that the probability is ½. This is not a situation of tossing a fair coin or dice. However, if God did indeed invent the integers, and God did invent Einstein, then maybe, God was playing dice with the universe and thus the assumption that the probability is ½ makes divine sense. In order to strengthen Crandall’s argument, another proof is necessary to prove that indeed, the probability of getting odd or even is ½. But we leave that to proof by divinity (the holy version). I think you are very close to a clear explanation of the probability argument. Can you patch it up?

Computer Proofs

Computer “proofs” involve checking a large, but finite number of cases by computer. In 1985, Yoneda at the University of Tokyo checked up to of x < 2^{40} [3]. Throughout the worldwide web, there are currently multi-site projects involving distributed computers trying to push the bounds of the problem. At first glance, this doesn’t appear to be a proof, in the classic sense. Also, in this way, it is impossible to prove that the trajectory will go on infinitely, as that requires infinite time, hardware, existence, etc. However, if one (or more precisely, many) can prove that a loop exists for x > 1, then we can power off our network and move on to the next unsolved problem in mathematics.

5 Conclusion
We have seen that a rather simple numeric function can generate interesting proofs that require deep analysis, ingenuity, and chance. In the history of mathematics, many problems initially deemed trivial have had lasting impact and/or were the springboard to other fascinating discoveries. Due to my mathematical immaturity, I could not delve into other areas such as the connections between the 3x+1 conjecture and ergodic theory or the more generalized version of the 3x+1 problem. But, as a consequence of writing this paper, I have developed an interest in graph theory and as my knowledge of graph theory grows, will most likely return to broach upon the 3x+1 conjecture with new lenses.

References


This is a well-written and interesting paper, but it lacks connections with contemporary research. You have only one paper after 2000, and you don’t seem to refer to it. You can only list papers in your bibliography that you reference. Can you get more information on the current state of research?

Problem statement 3/3

Current results 1/3

Important researchers 1/2

Why important and related problems 3/3

Organization 2/2

Use of language 4/4