

Twin Prime Conjecture

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ABSTRACT. This term paper will discuss the Twin Prime Conjecture. Many topics regarding the Twin Prime Conjecture will be cover: the preliminary definition of a Twin Prime Conjecture, the theorem, interest, mathematicians responsible for working with twin primes and related solved and/or unsolved problems. A conclusion incorporating logical strategy when faced with difficult theorems will the paper.

The twin prime conjecture is an unsolved mathematical question. It is unfortunately not one of the better known conjectures grand enough for the Millennium prize, but still a worthy conjecture to be explored. Defined twin primes, "are pairs of primes of the form $(p, p+2)$." (Weisstein) It was a German mathematician named Paul Stäckel (20 August 1862 – 12 December 1919) who was the first to use the term twin prime. (Wikipedia) I have not been able to find anything else to substantiate this claim.

Defining what twin primes are was the easy part. Trying to determine if there are infinitely many pairs of these prime twin pairs is the more challenging part. Looking at Stäckel's form for twin primes, $(p, p+2)$ we can easily see that by plugging in a prime integer the pair emerges. For example $(3, 5)$, $(5, 7)$, $(11, 13)$, $(17, 19)$, $(29, 31)$, $(41, 43)$... and so on...so we think. It is this unsolved question about twin primes that creates an environment where we cannot say for sure that these examples of twin prime pairs continue perpetually. Trying to find the proof that there are an infinite number of twin primes has been the work of many mathematicians over the years.

It would seem that the twin prime conjecture question has been circulating for decades, but after hearing that the question was asked before Paul Stäckel used the term twin prime makes me wonder. According to the PBS NOVA scienceNow podcast, the earliest known word on twin primes started with Euclid in 300b.c.e. (YouTube) From math enthusiasts to number theorists to even the more well know Dr. Daniel Goldston, mathematician, this mysterious unsolved question has required countless years of study to redefine our understanding of the conjecture.

The general public may have little interest in the Twin Prime Conjecture, but for a community of upper division math students, to doctoral mathematicians, the question, "Are there infinitely many twin primes?" (Wikipedia) sparks questions and hours of study to understand. This question has filled the pages of countless collegiate advanced math articles for decades and more recently, websites. These websites are dedicated to forum style discussions on the topic of the twin prime conjecture. Even the ever popular social media outlet facebook has a page discussing the twin prime conjecture.

In an article from *The American Mathematical Monthly*, written in 1960, author Golomb was working on finding a twin prime constant. The twin prime constant is the constant distribution of twin primes. He was using the prime number theorem, "[which]

Twin Prime Conjecture

gives an asymptotic form for the prime counting function.” (MathWorld.wolfram) to add in finding the probability of a twin prime constant. Golomb was researching the probability that, “Since x is prime, x is odd with probability 1, and $x + 2$ is odd, doubling the chance of $x + 2$ being prime.” (Golomb) Golomb’s article is an early description of what was surrounding the meat of the twin prime conjecture at that time.

15 years later, in *Mathematics of Computation*, an article written by Richard Brent, he explores the “Distribution of small gaps between successive primes.” Brent’s article makes use of many twin prime related unsolved conjectures, such as the Hardy and Littlewood conjecture. He uses the Hardy and Littlewood conjecture to substantiate his claims that given certain constraints he can, “[1], describe a method for computing these constraints [...] and [he was able to give] empirical evidence supporting the [Hardy and Littlewood] conjecture [...] mentioned.” (Brent) Brent was able to show that given constraints he could predict near accurately the gap distribution between small primes. In 1979, mathematicians Crandall and Penk take on the larger primes. In another article found in *Mathematics of Computation*, Crandall and Peck try to work out a way to find “Large twin prime pairs.” Crandall and Penk use asymptomatic estimates and primality tests to find large twin prime pairs. Crandall and Penk,

“define a twin prime mean (TPM) to be any positive integer m such that $(m - 1, m + 1)$ is a pair of primes.” [...] “by showing in effect that some positive constant C ,

$$M(x) < \frac{Cx}{\log^2 x} (\log \log x)^2$$

established that the sum of the reciprocals of all TPMs must converge. Twin prime pairs $(m - 1, m + 1)$ are, therefore, significantly sparser than the primes themselves, so the task of discovering large TPMs is both interesting and challenging.” (Crandall and Penk)

These mathematicians used the Hardy-Littlewood conjecture to support their estimations and then used primality tests to determine whether the TPM m can be located. Once located, Crandall and Penk use different tests and algorithms to find larger twin prime pairs. They note at the end of their findings, a point of interest for their reader. Crandall and Penk are quoted, “It is of interest that this statistical test gives an experimental value for C_2 which is within 2% of the Hardy-Littlewood prediction.”(387) It is these 2% errors in estimations that keep conjectures unproven.

For Crandall, Penk, and Brent, all men found what they were looking for. The 70’s seem to be a high time for twin primes. Doing further research I found that the interest in twin primes continues in the 90’s and more strongly into the current century, where Dr. Goldston has nearly proven the twin prime conjecture.

In January 1997, the hunt for large twin primes continues. Tony Forbes, author of the article, “A Large Pair of Twin Primes”, again from the journal *Mathematics of*

Twin Prime Conjecture

Computation, begins by summarizing his goal of finding a large twin prime using a “486 computer to discover a large pair of twin primes.” (Forbes) We intuitively can see that with the use of computers during this time becoming more frequent and normal, problems are getting easier to compute and ultimately easier to solve. Forbes was able to use known mathematical algorithms to successfully find what he is looking for. Forbes like Crandall and Penk ran a series of primality tests to find appropriate integers to plug into his chosen algorithm. Once tested, he tests again. In the end Forbes is able to print a list of very large twin prime numbers he can find. He concludes at the end of his article that if he was able to increase the speed of his computer, his tests would of concluded and gave results faster.

On Forbes suggestion of needing a faster computer 14 years ago, we now know that as computers get faster we are able to compute integers faster. We can plug integers into a formula programmed into a computer and see small twin primes and/or large twin primes as quickly as we want. Even with all this technology the unsolved question about twin primes still remains. Are there infinitely many?

In a more recent search on the internet I found a twin prime enthusiast named Tiebreaker. Tiebreaker typed up a very short proof for the twin prime conjecture and a clear example of what his proof looks like mathematically. Tie breakers proof claims that,

“As long as there is always a difference of 6 at some point between the two nearest odd non-primes, then there are infinitely many twin primes; because there can only be two odd numbers a distance of 2 from each other between them, and these must be prime since they are not non-prime but are odd.” [his example is as follows]
“There are no odd non-primes between 9 [6+3] and 15 [12+3] so 11 and 13 must be prime. Since the largest possible distance between all odd non-primes is 6 [Since 3 is the lowest odd number and its odd multiples are equal to $6x + 3$]
Then all twin primes can be written as $6x + 5$ and $6x + 7$.” (Yahoo)

Tiebreaker gives us much to think about. I can follow exactly what he has presented and I can find no flaws. Even though I cannot find any, I am skeptical to think that a person would just give a proof away like the unsolved twin prime conjecture to anyone who wanted to read it online. I emailed the one person who has recently made the most advancement towards proving the twin prime conjecture, Dr. Daniel Goldston a math professor at San Jose State University. Dr. Goldston’s advancement in nearly proving the twin prime conjecture has earned him great status among his fellow mathematicians. On the same NOVA scienceNow PBS podcast that said Euclid was the one who first mentioned twin prime, the program also talks about Daniel Goldston as being the closest to solving this unsolvable question. According to R. Miller,

“Two years ago, Daniel Goldston and his collaborators proved that there always exist primes that are very close together. With that success, Goldston hoped that their method would soon lead to a proof that there are infinitely often pairs of primes closer than some fixed bounded distance. Such a proof would be a giant step toward resolving the twin prime conjecture.” (Miller)

Twin Prime Conjecture

Dr. Goldston did not have time immediately to get back to me regarding progress made in proving the Twin Prime Conjecture, but he did have some interesting words on tiebreaker. Goldston writes, "... as for Tiebreaker: since each twin prime pair greater than 3 is of the form $6k-1$ and $6k+1$, it is clear that $6k-3$ and $6k+3$ will be nearest composite numbers (divisible by 3) that differ by 6. This doesn't actually tell you that such things exist infinitely often." (email 4-10-11) I had asked him if there has been any further advancement in proving this conjecture, and as mentioned, he originally did not have time, but recently found some.

Dr. Goldston emailed, "I might mention that when we suddenly broke through and found a method that answered some questions no one had realized how to do before, there was a period of about 2 months where we pretty much figured out everything, followed by another 4 months where we cleaned everything else up. Now I have spent 5 years trying to make the method work better and pretty much have gotten nowhere. But if you keep learning new things and get glimmers that something more can be done it is fun to keep at it. Hunting for results is what is interesting for me, not the results themselves." (Goldston email 4-20-11) Dr. Goldston also emailed me an unpublished article he is writing that will be published later this year. He discusses the recent progress made on the subject of twin primes, "In the case of twin primes, there have been several outstanding advances which to outward appearances could seem just as formidable as the twin prime conjecture. To mention just two of these results, it has been proved that for large enough x ,

$$\pi_2(x) \leq 8 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) x / \log^2 x$$

[...]we see that there can be at most 4 times as many twin primes as conjectured. Secondly, J. Chen proved that there are infinitely many primes p where $p + 2$ is either a prime or a product of two primes". (Goldston)

Goldston still finds himself working on trying to prove the Twin Prime Conjecture. Goldston and his team are making significant advancements and I am hopeful that soon they will have the breakthrough that they need to solve this centuries old question. To conclude with a quote from Dr. Goldston's paper, "You are welcome to try to prove this conjecture and become famous, but be warned that a great deal of effort has already been expended on this problem. The chances that a simple idea [...]will work is very small. Therefore also put some effort into understanding what has been learned about primes in the last two hundred years. (Goldston)

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Twin Prime Conjecture

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