The Collatz Conjecture: A Brief Overview

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The Collatz conjecture is an elusive problem in mathematics regarding the oneness of natural numbers when run through a specific function based on being odd or even, specifically stating that regardless of the initial number the series will eventually reach the number 1. This paper will describe the conjecture, explain the importance and intrigue of the problem, explore the different methods used in attempt to solve this problem over the years, and delve into why the problem has yet to be solved. The goal is to inform the reader on the nature of the conjecture, and hopefully spark an interest to attempt to solve this elusive conjecture.

A Brief Background

The Collatz Conjecture has been an internationally popular problem in mathematical circles since the early part of the 20th century when the German mathematician Lothar Collatz is credited with the origination of the problem. While the vast majority of lay documents concerning the Collatz Conjecture state that Lothar Collatz first proposed it in 1937, the author of this paper could find no solid citation to back up this claim. The testimonies of his students as well as his own notebooks document Collatz working problems of a similar nature as early as 1932 (Lagarias, 2010). Collatz, himself, lays no claim to the problem, but does state that he discussed and explained the problem to H. Hasse in 1952 (Collatz, 1986) For the most part due to a complete lack of relevant results, nothing was published about the problem until 1971 when a lecture by H.S.M. Coxeter was put into print (Lagarias, 2010). In his paper titled "Cyclic Sequences and Frieze Patterns" Coxeter states the problem explicitly as a "more recent piece of mathematical gossip" and offers no attempt at a solution. Bryan Thwaites is said to have also laid claim to the origination of the problem in 1952, but Collatz is widely considered to be the true originator. Even still, "On the motivation and origin of the (3n+1) problem" was not put out until 1986, and remains the only paper ever published on the problem by Collatz himself.

While iterations have tested positive through $x < 5.7646 * 10^{18}$ (Lagarias 2010), an adequate proof has yet to be presented and it is widely considered to be an extremely difficult problem. At the end of Coxeter's original paper discussing the problem in 1971, he gives a caveat, followed with some much needed moral support for anyone in search of a solution:

> "I must warn you not to try this in your heads or on the back of an old envelope, because the result has been tested with an electronic computer for all $x_1 \leq 500{,}000$. This means that, if the conjecture is false, the prizewinner muse either find a sequence of this kind which he can prove to be divergent, or else find a cyclic sequence for this kind whose terms are all greater than half a million.
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> "Whenever you spend days or weeks struggling with a problem that comes to nothing, think of poor Sisyphos. As Felix Behrend puts it at the end of his book, "Sisyphos and his stone are the symbol of man and his eternal striving, never ceasing, never fulfilled, and yes always triumphant. What more can you ask?""

The eccentric Hungarian mathematician Paul Erdős claimed that "Mathematics is not yet ready for such problems," and referred to the conjecture as "Hopeless. Absolutely hopeless."

The Problem Itself

The Collatz Conjecture describes the iterations of integers applied to a very simple function. The conjecture specifically states: "Starting from any positive integer *n*, iterations of the function *C(x)* will eventually reach the number 1. Thereafter iterations will cycle taking successive values 1, 4, 2, 1, ..." (Lagarias, 2010).

To define a basic term, an integer x will be defined as odd when $x \equiv 1 \pmod 2$. Likewise, x will be defined as even when $x \equiv 0 \pmod 2$. With those common terms specified, the following is the function known as the Collatz function:

$$C(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ \frac{1}{2} * x & \text{if } x \text{ is even} \end{cases}$$

The Collatz function is named as such with respect to it's originator. However, for the purpose of analysis, a more succinct function describes the same graph with fewer iterations, as the odd component of the function, $C(x) = 3x+1$, ensures that the following iteration will result in an even value. This function, supported by C.J. Everett in "Iteration of the number-theoretic function: $f(2n) = n$, $f(2n+1) = 3n+2$" is as follows:

$$T(x) = \begin{cases} \frac{1}{2} * (3x+1) & \text{if } x \text{ is odd} \\ \frac{1}{2} * x & \text{if } x \text{ is even} \end{cases}$$

The function $T(x)$ is logically related to $C(x)$ in that the odd component of $T(x) = C(C(x))$, while the even component remains the same. While research is far more common on this version of the function than the original, their relation makes the research directly relevant.

The allure of the problem and frustration of many mathematicians is the seemingly predictable randomness of the iterations. The specific number of iterations it takes for a starting value to reach 1 is referred to as the "total stopping time." The total stopping time is a very important value, as it is the point of focus for much of the research done on the Conjecture. As an example of this tantalizing randomness, the recorded total stopping time of initial values that are relatively close to one another seem to form patterns, but in the end are random. Often two or thee initial values in a row have the same stopping time, but in a completely unpredictable way. The initial values 1004, 1005, and 1006 all have total stopping times of 45, for example. Also, initial values in the range of 1000-1099, only nineteen total stopping times exist, with the total stopping times of 23 and 80 appearing 17 and 16 times respectively. Tendencies of the total stopping time can be loosely mapped, but ever so loosely. Iteration cycles have been studied at depth, but no avenue of research has proved fruitful in the search of a proof.

Importance & Related Problems

While the problem itself remains unsolved and seemingly unapproachable, a fair amount of research has been done on the generalizations of the problem when viewed as a specific case of a more general class of functions. Some of these more general functions are analyzable. Such generalized $3x+1$ problems include the "3x+d" problem which showed that all integer orbits are eventually periodic for d ≥ -1 (Belaga and Mignotte , 1998), and the "qx+1" problem which showed that problems of similar structure can indeed be proven (Steiner, 1981). These results provide a plausible model for the specific $3x+1$ problem, but do not necessarily approach a solution.

There are a many factors that contribute to the overall difficulty of the problem. Pseudorandomness, one of a few major influences on the difficulty and elusiveness of the Collatz Conjecture, is related to ergodic theory which is beyond the technical scope of this overview. However, according to Lagarias, the connection shows that "the iterates of the shift function are completely unpredictable in the ergodic theory sense." This pseudorandomness can be observed for all values of x until $x = 2^n$ for any positive integer n. "This supports the $3x+1$ conjecture and at the same time deprives us of any obvious mechanism to prove it, since mathematical arguments exploit the existence of structure, rather than its absence."

Another issue, which is described in depth in J.H. Conway's paper "Unpredictable Iterations", deals with the inability of the any sort of computer generated algorithm to predict nearly anything about the iterations in the long run. This roadblock which Conway refers to as "non-computability" reveals that the problem could indeed be unsolvable, and a method to approach the issue is unknown (Lagarias, 2010).

From the perspective of an individual less applauded in the field, Peter Schorer of Hewlett-Packard Laboratories claims that "one reason the problem is so difficult is that (informally) the structure of counterexamples to the $3x+1$ Conjecture, and the structure of non-counterexamples, are so similar. For example, the inverse of each range element y of the $3x+1$ function, be that range element a counterexample or a non-counterexample, is an infinitary tree with y as a root. Furthermore, all the properties of these trees that we are aware of, are the same regardless whether the root is a counterexample or a non-counterexample." Schorer then proceeds to attempt to prove the conjecture by showing that there is no difference between counterexample tuples and non-counterexample tuples. His proof has yet to gain any wide acceptance.

The known difficulty of the problem as well as the seemingly simple nature of the function has lead to research in many different fields, namely number theory, dynamical systems, computer science, ergodic theory, probability theory, and computational theory. R.K. Guy worked on the problem from a number theory perspective on the connection that the problem is arithmetic in nature. Classes of generalized versions on the function have been defined under certain conditions. In dynamical systems, the problem is studied via the behavior of the function under iteration. Computational and Fractran models have been used to show the validity of the conjecture to a very large scale in the computer science field. Ergodic theory deals with the presence of an invariant measure in the dynamic system. Probability theory attempts to model the behavior of the iteration. Lastly, computational theory connects with the Collatz Conjecture via J.H. Conway, who states that "there is a generalized $3x+1$ function whose iteration can simulate a universal computer." (Lagarias, 2010). The fact that the Collatz Conjecture spreads across so many different fields of mathematics has allowed many great minds to work on and contribute to the knowledge base of the problem. It has opened up avenues of research in all of these disciplines and has lead to some important results outside of the conjecture itself.

To the avail of many a mathematician, in spite of the results generated by supercomputers, and mocking the analysis of generalized forms of the function, the Collatz Conjecture remains unsolved and seems to be unsolvable. A fair amount is known about it, but the vast majority of that knowledge has proved useless in the realm of proving the conjecture. The broad scope of the problem, its seemingly simple nature, and the vast depth of related problems will continue to intrigue and puzzle mathematicians for, quite possibly, a very long time to come.

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