Primer on Mortgages

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1 Fixed Rate Mortgages

From the borrower's point of view, a fixed payment loan is characterized by PV - the sum of the loan (present value), r - periodic interest rate, T - number of payments, and PMT - periodic payment. The fixed periodic payment is calculated by solving

$$PV = \sum_{t=1}^{T} \frac{PMT}{(1+r)^t} = PMT \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^t = PMT \cdot R(r,T)$$

Notice that the periodic payment is proportional to the sum of the loan, and the constant of proportionality is given by $R(r,T)$, which depends on the periodic interest rate and the number of payments. Using the summation formula in the appendix, with $q = 1/(1+r)$ we have for $r \neq 0$

$$R(r,T) = \sum_{t=1}^{T} q^t = \frac{q - q^{T+1}}{1-q} = \frac{\left(\frac{1}{1+r}\right) - \left(\frac{1}{1+r}\right)^{T+1}}{1 - \left(\frac{1}{1+r}\right)} = \frac{1 - \left(\frac{1}{1+r}\right)^T}{r}$$

Thus

$$R(r,T) = \begin{cases} \left[1 - \left(\frac{1}{1+r}\right)^T\right]/r & \text{if } r \neq 0 \\ T & \text{if } r = 0 \end{cases}$$

The constant payment is then

$$PMT = PV/R(r,T)$$

Example 1. Suppose I take a mortgage of $\$720,000$ for 30 years, with annual interest rate of 5% and monthly payments. Calculate the monthly payment. Thus, $PV = \$720,000$, monthly interest rate is $r = 5\%/12$, the number of monthly payments is $T = 12 \cdot 30 = 360$.

$$\begin{aligned} R &= \frac{1 - \left(\frac{1}{1+r}\right)^T}{r} = \frac{1 - \left(\frac{1}{1+0.05/12}\right)^{360}}{0.05/12} = 186.281617 \\ PMT &= PV/R = \$3865.12 \end{aligned}$$

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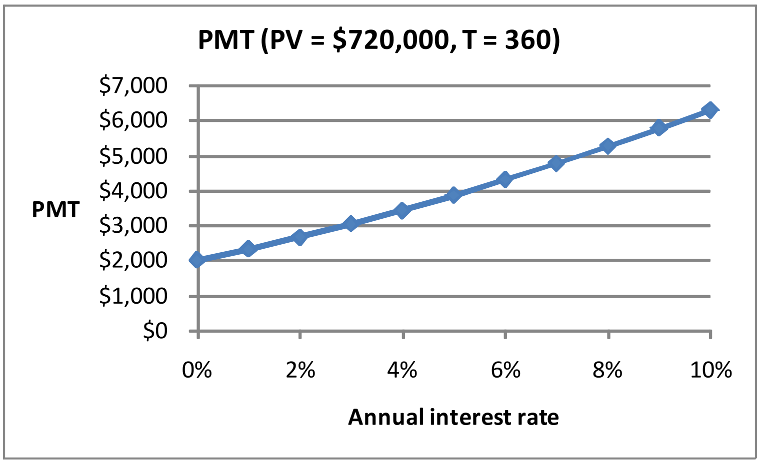
Example 2. Plot the graph of PMT for the above loan, for various annual interest rates: $r = 0\%, 1\%, ..., 10\%$.



Next, we would like to break the monthly payment into interest payment, INT, and principal payment, PAL. This is important because interest payments on mortgages are tax deductible. This is done by calculating the balance after any number of payments. Let $BAL(n)$ be the remaining balance on the mortgage after n payment. For example, $BAL(20)$ is the remaining balance after you have already made 20 payments. Knowing the remaining balance is important because this is the amount that you need to borrow after n payments if you wish to *refinance* the loan.

After the first payment the balance is equal to

$$BAL(1) = PV(1+r) - PMT$$

Thus, the balance accumulates due to interest and declines due to your payment. Similarly, the balance after the second payment is

$$\begin{aligned} BAL(2) &= BAL(1)(1+r) - PMT \\ &= [PV(1+r) - PMT](1+r) - PMT \\ &= PV(1+r)^2 - PMT(1+r) - PMT \end{aligned}$$

The balance after the third payment is

$$\begin{aligned} BAL(3) &= BAL(2)(1+r) - PMT \\ &= \left[PV(1+r)^2 - PMT(1+r) - PMT\right](1+r) - PMT \\ &= PV(1+r)^3 - PMT(1+r)^2 - PMT(1+r) - PMT \end{aligned}$$

We can keep iterating like that to find the balance on the loan after n payments. In general, the closed form solution to $BAL(n)$ is

$$\begin{aligned} BAL(n) &= PV(1+r)^n - PMT\sum_{t=0}^{n-1}(1+r)^t \\ &= PV(1+r)^n - PMT\left(\frac{1-(1+r)^n}{1-(1+r)}\right) \\ &= PV(1+r)^n + PMT\left(\frac{1-(1+r)^n}{r}\right) \end{aligned}$$

Of course, $BAL(0) = PV$, since if you did not make any payments yet, your balance is the amount of the loan.

Example 3. For the mortgage in example 1, calculate the remaining balance after 20 payments.

$$\begin{aligned} BAL(20) &= PV(1+r)^n + PMT\left(\frac{1-(1+r)^n}{r}\right) \\ &= 720,000\cdot(1+5\%/12)^{20} + \$3865.12\left(\frac{1-(1+5\%/12)^{20}}{5\%/12}\right) \\ &= \$701,995.37 \end{aligned}$$

Check that $BAL(360) = 0$.

The interest component of payment number n is the interest that you pay on the balance remaining after $n-1$ payments. For example, the interest component of the first payment is the interest accumulated on the entire loan during the first month. In general, the interest component of payment number n is given by

$$INT(n) = r\cdot BAL(n-1)$$

Example 4. What is the interest part of the 21st payment in the last example? What is the principal part of the 21st payment?

$$INT(n) = (5\%/12)\cdot BAL(20) = (5\%/12)\cdot\$701,995.37 = \$2,924.98$$

This means that if you are in the 25% federal income tax bracket, you essentially getting a tax discount (a subsidy) from the federal government of $25\%\cdot\$2,924.98 = \$731.25$ for that payment. The principal part is the remaining portion of the periodic payment, which is not interest payment. In the above example

$$PAL(n) = PMT - INT(n) = \$3865.12 - \$2,924.98 = \$940.13$$

Summary. Now you know everything there is to know about fixed interest loans. In particular, you know how to calculate the periodic payment, the balance on the loan after any number of payments, and what is the interest part and principal part of each payment. It is useful to construct a table in Excel spreadsheet that specifies all the payments, broken into interest and principal components. Within the same spreadsheet you can also calculate the tax savings on the interest. The next table demonstrates such mortgage payments plan, when I skip most of the table, and present the beginning and the end only.

n	BAL	int	PMT	Principle
0	$720,000.00			
1	$719,134.88	$3,000.00	$3,865.12	$865.12
2	$718,266.16	$2,996.40	$3,865.12	$868.72
3	$717,393.82	$2,992.78	$3,865.12	$872.34
4	$716,517.85	$2,989.14	$3,865.12	$875.97
5	$715,638.22	$2,985.49	$3,865.12	$879.62
6	$714,754.93	$2,981.83	$3,865.12	$883.29
7	$713,867.96	$2,978.15	$3,865.12	$886.97
⋮				
353	$26,610.46	$126.45	$3,865.12	$3,738.66
354	$22,856.22	$110.88	$3,865.12	$3,754.24
355	$19,086.34	$95.23	$3,865.12	$3,769.88
356	$15,300.75	$79.53	$3,865.12	$3,785.59
357	$11,499.39	$63.75	$3,865.12	$3,801.36
358	$7,682.18	$47.91	$3,865.12	$3,817.20
359	$3,849.08	$32.01	$3,865.12	$3,833.11
360	$0.00	$16.04	$3,865.12	$3,849.08

Notice that over time, the interest part on the loan declines, since the balance declines and interest is paid on the remaining balance. Thus, towards the end of your mortgage payments, the tax discount declines as well.

2 Variable Rate Mortgages

A variable interest mortgage, forces the borrower to refinance each time the interest rate changes. In particular, in the context of *subprime mortgage crisis*, the initial interest rate is low for given amount of time, and after that the interest rate increases. We will look at a loan that charges interest rate r_1 for n periods, and then charges interest rate r_2 for the remaining $T-n$ periods. We need to calculate two payments now, PMT_1 which is paid during periods $1, 2, ..., n$ and PMT_2 which is paid during periods $n+1, n+2, ..., T$. There is nothing new in terms of the math involved in these calculations. The first payment is computed as if you take a loan at fixed interest r_1 for the entire duration of the loan. The second payment is computed as if you are taking a new loan at the size of the remaining balance after n periods, but the new loan is charging interest rate r_2.

Formally, the first payment is given by

$$\begin{aligned} R_1(r_1, T) &= \begin{cases} \left[1 - \left(\frac{1}{1+r_1}\right)^T\right]/r_1 & \text{if } r_1 \neq 0 \\ T & \text{if } r_1 = 0 \end{cases} \\ PMT_1 &= PV/R(r_1, T) \end{aligned}$$

The second payment, after the change in interest rate, is given by

$$\begin{aligned} R_2\left(r_2, T-n\right) &= \begin{cases} \left[1-\left(\frac{1}{1+r_2}\right)^{T-n}\right]/r_2 & \text{if } r_2 \neq 0 \\ T & \text{if } r_2 = 0 \end{cases} \\ PMT_2 &= BAL\left(n\right)/R\left(r_2, T-n\right) \end{aligned}$$

Example 5. Consider example 3 again, but now the interest rate after 20 payments jumps from 5% to 9%. What is the new monthly payment? Recall that the balance after 20 payments is $\$701,995.37$. Now that the interest rate has changed, the borrower is forced to *refinance* this amount at higher interest rate of 9%. We can treat this change as taking a new loan of $\$701,995.37$ at higher interest rate and duration of $360-20=340$ payments. Using the above formula, we get

$$\begin{aligned} R_2\left(r_2, T-n\right) &= \frac{\left[1-\left(\frac{1}{1+r_2}\right)^{T-n}\right]}{r_2} = \frac{\left[1-\left(\frac{1}{1+0.09/12}\right)^{340}\right]}{0.09/12} = 122.82 \\ PMT_2 &= \frac{BAL\left(n\right)}{R\left(r_2, T-n\right)} = \frac{\$701,995.37}{122.82} = \$5715.51 \end{aligned}$$

3 Appendix: Summation Formula

Theorem 1 *Let $q \neq 1$. Then,*

$$\sum_{t=n}^{T} q^t = \frac{q^n - q^{T+1}}{1-q}$$

If $q = 1$, then

$$\sum_{t=n}^{T} q^t = \sum_{t=n}^{T} 1 = T + 1 - n$$

Proof.

$$\sum_{t=n}^{T} q^t = q^n + q^{n+1} + ... + q^{T-1} + q^T$$

Multiply by $1-q$

$$q^n + q^{n+1} + ... + q^{T-1} + q^T - \left(q^{n+1} + q^{n+2} + ... + q^T + q^{T+1}\right) = q^n - q^{T+1}$$

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