Online Appendix for: Why did Rich Families Increase their Fertility? Inequality and Marketization of Child Care

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Abstract

In this Online Appendix we first perform robustness exercises on Figure 5 of the main paper. We then present the proofs referenced in the main paper, specifically showing the existence and uniqueness of the solution to the household problem and the shape of fertility across income deciles. We then include our robustness test allowing for the costs and returns to education to change over time. Finally, we discuss the normalization of parameters.
A Cross State Relationship Between the Relative Price of Marketization and High Income Fertility

In this section we explore further the cross-state relationship between changes in high income fertility between 1980 and 2010 and the change in the relative price of marketization, as first introduced in Figure 5 paper.

We estimate regressions of the following structure:

\[
\Delta \% n_s = \Delta \% \left( \frac{w_{fs}}{w_{HPS}} \right) + \Delta \% w_{ms} + \delta_r + \epsilon_s,
\]

where the dependent variable \( \Delta \% n_s \) is the percentage change in hybrid fertility rates for the top two decile of white non-Hispanic married women in state \( s \). The main explanatory variable of interest, \( \Delta \% \left( \frac{w_{fs}}{w_{HPS}} \right) \), is the percentage change in the ratio of the average wage of white non-Hispanic married women in the top two deciles to the average wage in the home production substitute sector in state \( s \). \( \Delta \% w_{ms} \) is the percentage change of the average wage of white non-Hispanic married men in the top two deciles in state \( s \). \( \delta_r \) is a set of region fixed effects for each region \( r \in \{\text{Northeast, South, Midwest, West}\} \). \( \epsilon_s \) is an error term. These variables are described in detail in Appendix A of the main paper. All regressions are estimated with robust standard errors.

\( \Delta \% \left( \frac{w_{fs}}{w_{HPS}} \right) \) captures the change, over our time period, in the relative price of marketizing a woman’s time. Quantitatively, this variable is shown to be crucial for explaining changing fertility patterns in Section 4.4 of the main paper. \( \Delta \% w_{ms} \) captures changes in the demand for children induced by increases in male wages and quantitatively evaluated in Section 4.4 of the main paper. The regional fixed effects are implicitly interacting differentially with time, as all our variables are changes between 1980 and 2010. This allows us to control for differential regional trends.

Table 1 describes the results. Column 1 regresses changes in fertility only on changes in the relative price of marketization. Notice that this regression describes the results shown graphically in Figure 5 of the main paper. Column 2
Table 1: The Effect of Marketization on High-Income Fertility

<table>
<thead>
<tr>
<th></th>
<th>Dependent Variable: Percent Change in High-Income Fertility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>( \Delta % \left( \frac{w_f}{w_i} \right) )</td>
<td>1.064***</td>
</tr>
<tr>
<td></td>
<td>(0.279)</td>
</tr>
<tr>
<td>( \Delta % w_{ms} )</td>
<td>-0.916</td>
</tr>
<tr>
<td></td>
<td>(1.453)</td>
</tr>
<tr>
<td>Region FE</td>
<td>No</td>
</tr>
<tr>
<td>Obs.</td>
<td>50</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.154</td>
</tr>
</tbody>
</table>

Notes: Robust standard errors in parentheses. * \( p < 0.10 \), ** \( p < 0.05 \), *** \( p < 0.01 \)

adds the change in male wages over time, while Column 3 adds region fixed effects. Columns 4–6 repeat Columns 1–3, but drop outlying observations, namely North Dakota and Wyoming.

All specifications show a statistically significant and economically meaningful elasticity between fertility and the relative price of marketization, of around 1. Controlling for changes in men’s wages or regional fixed effects do not have meaningful effects on the point estimates or standard errors. Dropping outliers strengthens all specifications, but also increases the standard errors somewhat. In particular, the estimate in Column 6 has a point estimate that is basically the same as its counterpart in Column 3, but has a higher p-value of 0.053. All other specifications are significant at the 1% level. Changes in men’s wages do not have a meaningful impact on changes in fertility rates, consistent with their relatively weak effect in the model, documented in Section 4.4 of the main paper.

B Proofs

B.1 Existence and Uniqueness of the Solution to the Household Problem

Proposition 1 The necessary and sufficient condition for existence of a unique solution to the household’s problem is \( \frac{\rho \sigma}{\alpha} < 1 \).
Proof. The household’s optimization problem can be written as follows:

$$\max_{e \geq 0} U(e) = -\ln \left( \frac{p_n}{p_e} + e \right) + \frac{\beta \theta}{\alpha} \ln (e + \eta)$$

There is a possibility that $U(e)$ is unbounded above, and therefore the household’s problem has no solution. We can write the objective function as follows:

$$U(e) = \ln \left( \frac{(e + \eta)^{\frac{\beta \theta}{\alpha}}}{\frac{p_n}{p_e} + e} \right)$$

Taking the limit as $e \to \infty$,

$$\lim_{e \to \infty} U(e) = \ln \left( \lim_{e \to \infty} \frac{(e + \eta)^{\frac{\beta \theta}{\alpha}}}{\frac{p_n}{p_e} + e} \right) = \ln \left( \lim_{e \to \infty} \frac{\frac{\beta \theta}{\alpha} (e + \eta)^{\frac{\beta \theta}{\alpha} - 1}}{1} \right) = \begin{cases} \infty & \frac{\beta \theta}{\alpha} > 1 \\ 1 & \frac{\beta \theta}{\alpha} = 1 \\ -\infty & \frac{\beta \theta}{\alpha} < 1 \end{cases}$$

The first step used chain rule of limits, and the second step used L’Hospital’s rule since we have a limit of the form $\frac{\infty}{\infty}$. Intuitively, $\frac{\beta \theta}{\alpha}$ is the weight on quality in the utility function. When this weight is very high, it is possible that the household would like to choose $e \to \infty$ and $n \to 0$, which makes the problem unsolvable. Thus, in order to make the objective function bounded above, we have to impose the restriction $\frac{\beta \theta}{\alpha} \leq 1$.

Case 1: $\frac{\beta \theta}{\alpha} = 1$

$$U(e) = -\ln \left( \frac{p_n}{p_e} + e \right) + \ln (e + \eta)$$

$$U'(e) = -\frac{1}{\frac{p_n}{p_e} + e} + \frac{1}{e + \eta}$$
In this case, the solution to the household’s problem is as follows:

\[ \frac{p_n}{p_e} > \eta \Rightarrow U'(e) > 0 \quad \forall e, \text{ i.e. } U(e) \text{ is monotone increasing, } e^* \to \infty \]

\[ \frac{p_n}{p_e} < \eta \Rightarrow U'(e) < 0 \quad \forall e, \text{ i.e. } U(e) \text{ is monotone decreasing, } e^* = 0 \]

\[ \frac{p_n}{p_e} = \eta \Rightarrow U'(e) = 0 \quad \forall e, \text{ i.e. } U(e) \text{ is constant, } e^* \in (-\infty, \infty) \]

Case 2: \( \frac{\beta \theta}{\alpha} < 1 \)

In this case, the first order necessary condition for interior maximum is \( U'(e^*) = 0 \):

\[
U(e) = -\ln \left( \frac{p_n}{p_e} + e \right) + \frac{\beta \theta}{\alpha} \ln (e + \eta)
\]

\[
U'(e) = -\frac{1}{\frac{p_n}{p_e} + e} + \frac{\beta \theta}{\alpha} \frac{1}{e + \eta} = 0
\]

\[
\frac{\eta + e}{\frac{p_n}{p_e} + e} = \frac{\beta \theta}{\alpha}
\]

\[
e^* = \frac{\beta \theta \frac{p_n}{p_e} - \eta}{1 - \frac{\beta \theta}{\alpha}}
\]

The second order sufficient condition for \( e^* \) to be a local maximizer is:

\[
\frac{1}{(\frac{p_n}{p_e} + e^*)^2} - \frac{\beta \theta}{\alpha} \frac{1}{(\eta + e^*)^2} < 0
\]

\[
\frac{(\eta + e^*)^2}{(\frac{p_n}{p_e} + e^*)^2} < \frac{\beta \theta}{\alpha}
\]
Using the first order condition:

\[
\frac{\beta \theta}{\alpha} < \frac{\beta \theta}{\alpha} \quad \text{and} \quad \frac{\beta \theta}{\alpha} < 1
\]

Thus, \( \frac{\beta \theta}{\alpha} < 1 \) guarantees that a solution to the household’s problem exists, and the first order necessary condition is a local maximum. Moreover, since the critical point is unique, the local maximum must also be the unique global maximizer.

**B.2 Shape of fertility across income deciles**

We start with preliminary derivations needed for the proofs below. We focus on the region of parameter values where the solution is interior, i.e. \( e^* > 0 \). In this case, optimal fertility is given by

\[
e^* = \max \left\{ \frac{p_n \frac{\beta \theta}{\alpha} - \eta}{1 - \frac{\beta \theta}{\alpha}}, 0 \right\}
\]

(2)

\[
n^* = \left( 1 - \frac{\beta \theta}{\alpha} \right) \frac{\alpha}{1 + \alpha} \left( \frac{w_f + w_m}{p_n - \eta p_e} \right)
\]

(3)

where

\[
p_n = \frac{1}{A} \left[ \phi^{\frac{1}{n}} w_f^{\frac{p}{n-1}} + (1 - \phi)^{\frac{1}{n}} p_m^{\frac{p}{n-1}} \right]^{\frac{n-1}{p}}.
\]

(4)

Notice that \( e^* > 0 \) and existence of solution to household’s problem, \( \frac{\beta \theta}{\alpha} < 1 \), imply that \( n^* > 0 \) and \( p_n - \eta p_e > 0 \). To analyze the effects on fertility, it suffices to ignore the constant term and focus on the ratio term \( \frac{w_f + w_m}{p_n - \eta p_e} \). Clearly, \( n^* \) is increasing in \( w_m \) as male wages work purely through the positive income effect appearing in the numerator. Female wages, however, affect both the numerator (the positive income effect) and the denominator (the negative price effect). Let \( \mathcal{E}_{Y,X} \) denote the elasticity of \( Y \) with respect to \( X \). It follows that, for small percentage changes in \( w_m \) and \( w_f \), the approximate implied change in \( n^* \) is given
by

\( \% \Delta n^* \approx \varepsilon_{num,w_m} \% \Delta w_m + \varepsilon_{num,w_f} \% \Delta w_f - \varepsilon_{denom,w_f} \% \Delta w_f \) 

where the elasticity terms are computed as follows:

\[
\begin{align*}
\varepsilon_{num,w_f} &= \frac{\partial (w_f + w_m)}{\partial w_f} \frac{w_f}{w_f + w_m} = \frac{w_f}{w_f + w_m} \\
\varepsilon_{num,w_m} &= \frac{\partial (w_f + w_m)}{\partial w_m} \frac{w_m}{w_f + w_m} = \frac{w_m}{w_f + w_m} \\
\varepsilon_{denom,w_f} &= \frac{\partial (p_n - \eta p_e)}{\partial w_f} \frac{w_f}{p_n - \eta p_e} = \varepsilon_{p_n,w_f} \frac{p_n}{p_n - \eta p_e}
\end{align*}
\]

and

\( \varepsilon_{p_n,w_f} = \frac{\partial p_n w_f}{\partial w_f p_n} = \frac{\phi^{\frac{1}{1-\varphi}} w_f^\varphi}{\phi^{\frac{1}{1-\varphi}} w_f^\varphi + (1 - \phi)^{\frac{1}{1-\varphi}} p_m^\varphi} \in (0, 1). \)

The question is how optimal fertility varies across couples that represent different income deciles for a given year, or the same decile across years. These couples differ on \( w_m \) and \( w_f \). From (5) we see that for \( n^* \) to decline across income deciles in 1980, as was observed in the data, the price effect of \( \% \Delta w_f \) must dominate the income effect of both \( \% \Delta w_f \) and \( \% \Delta w_m \), where the \( \% \Delta \)'s are taken across consecutive income deciles. Moreover, for \( n^* \) to increase between 1980 and 2010 for couples representing high income deciles, the price effect due to \( \% \Delta w_f \) must yield to the income effect due to both \( \% \Delta w_f \) and \( \% \Delta w_m \). In this case, the \( \% \Delta \)'s refer to changes over time for a fixed decile. Because the effect of \( w_m \) on \( n \) is always positive (\( \varepsilon_{num,w_m} > 0 \)), we focus on investigating the effect due to \( w_f \) alone (\( \varepsilon_{num,w_f} - \varepsilon_{denom,w_f} \)). This is done in the two propositions to follow. However, bear in mind that to understand the profile of optimal fertility across income deciles or over time, we need to consider the combined effects of both \( w_m \) and \( w_f \).

**Proposition 2.** (Monotonicity and limit of \( \partial n^*/\partial w_f \)). If \( \rho \in (0, 1) \), i.e. inputs in
the home production are substitutes, then (a) \( \partial n^*/\partial w_f \) is monotonically increasing in \( w_f \) and (b) strictly positive for a large enough \( w_f \), i.e. \( \lim_{w_f \to \infty} \partial n^*/\partial w_f > 0 \).

**Proof.** Proof of (a). Differentiating (3) with respect to \( w_f \) and omitting the positive constant term gives

\[
\frac{\partial n^*}{\partial w_f} \propto \frac{(p_n - \eta p_e) - (w_f + w_m) \frac{\partial p_n}{\partial w_f}}{(p_n - \eta p_e)^2} = \frac{1 - \left(\frac{w_f + w_m}{w_f}\right) \mathcal{E}_{p_n,w_f} \frac{p_n}{p_n - \eta p_e}}{p_n - \eta p_e}.
\]

The denominator is positive. To show that the ratio is monotonically increasing, it suffices to show that the negative term in the numerator is made up of positive and monotone decreasing functions of \( w_f \). This is seen from obtaining a negative derivative for each of the product terms:

\[
\frac{\partial}{\partial w_f} \left(\frac{w_f + w_m}{w_f}\right) = -\frac{w_m}{w_f^2} < 0,
\]

\[
\frac{\partial}{\partial w_f} \mathcal{E}_{p_n,w_f} = \frac{\left(\frac{\rho \phi^{1-\rho}}{p_{n-1}} w_f \right)^{-1}}{\left(\phi^{1-\rho} w_f^{-\rho} + (1 - \phi) \phi^{1-\rho} p_m^{-\rho}\right)^2} < 0, \text{ when } \rho \in (0, 1),
\]

and

\[
\frac{\partial}{\partial w_f} \left(\frac{p_n}{p_n - \eta p_e}\right) = \frac{-\eta p_e \frac{\partial}{\partial w_f} p_n}{(p_n - \eta p_e)} < 0,
\]

where the last inequality follows from showing that \( \frac{\partial}{\partial w_f} p_n > 0 \), as can be seen from Equation (4).

(b) Proof of (b). Because the limit of a product of functions is equal to the product of limits, we obtain

\[
\lim_{w_f \to \infty} \left(\frac{w_f + w_m}{w_f}\right) \mathcal{E}_{p_n,w_f} \frac{p_n}{p_n - \eta p_e} = \lim_{w_f \to \infty} \left(\frac{w_f + w_m}{w_f}\right) \cdot \lim_{w_f \to \infty} \mathcal{E}_{p_n,w_f} \cdot \lim_{w_f \to \infty} \frac{p_n}{p_n - \eta p_e}.
\]
Each limit can then be obtained. First,
\[
\lim_{w_f \to \infty} \left( \frac{w_f + w_m}{w_f} \right) = 1.
\]

Second,
\[
\lim_{w_f \to \infty} E_{p_n,w_f} = \lim_{w_f \to \infty} \frac{\phi^{\frac{1}{\rho}}}{\phi^{\frac{1}{\rho}} + (1 - \phi) \frac{1}{\rho} \left( \frac{p_m}{w_f} \right)^{\frac{\rho-1}{\rho}}} = 0,
\]
which follows from \( \lim_{w_f \to \infty} \left( \frac{p_m}{w_f} \right)^{\frac{\rho}{\rho-1}} = \infty \) implied by the assumption that \( \rho \in (0,1) \). To derive the final limit, we first note that
\[
\lim_{w_f \to \infty} p_n = \lim_{w_f \to \infty} \frac{1}{A} \left[ \phi^{\frac{1}{\rho}} w_f^{\frac{\rho}{\rho-1}} + (1 - \phi) \frac{1}{\rho} p_m^{\frac{\rho}{\rho-1}} \right]^{\frac{\rho-1}{\rho}} = \frac{1}{A} (1 - \phi)^{-\frac{1}{\rho}} p_m > 0
\]
whenever \( \rho \in (0,1) \). It follows that the third limit is
\[
\lim_{w_f \to \infty} \frac{p_n}{p_n - \eta p_e} = \frac{1}{A} (1 - \phi)^{-\frac{1}{\rho}} p_m - \eta p_e \in (1, \infty).
\]
The product of these three limits is zero. It follows that
\[
\lim_{w_f \to \infty} \frac{\partial n^*}{\partial w_f} \propto \lim_{w_f \to \infty} \frac{1 - \left( \frac{w_f + w_m}{w_f} \right) E_{p_n,w_f} \frac{p_n}{p_n - \eta p_e}}{p_n - \eta p_e} = \frac{1}{(1 - \phi)^{-\frac{1}{\rho}} p_m - \eta p_e} > 0.
\]

**Corollary 1** In the region of \( w_f \) where the solution is interior, \( n^* \) is either U-shaped or monotonically increasing in \( w_f \).

**Proof.** Observe from (2) that \( e^* \) is monotone increasing in \( w_f \) whenever \( e^* \) is interior. Thus, there exists a well-defined lowest wage marking the interior solution \( w_f^* \equiv \inf \{ w_f | e^* (w_f) > 0 \} \). If \( \partial n^* / \partial w_f (w_f) \geq 0 \), i.e. \( n^* \) is non-decreasing near \( w_f^* \), then we know by proposition 2 that \( \partial n^* / \partial w_f > 0 \) for larger \( w_f \). In this case,
\( n^* \) is strictly increasing in the region of \( w_f \geq \overline{w_f} \). If, however, \( \partial n^*/\partial w_f (w_f) < 0 \), i.e. \( n^* \) is decreasing in \( w_f \) near \( \overline{w_f} \), then we know by proposition 2 that \( \partial n^*/\partial w_f \) will monotonically increase with \( w_f \) becoming strictly positive for a large enough \( w_f \). In this case, \( n^* \) is a U-shape function of \( w_f \) in the region of \( w_f \geq \overline{w_f} \).

## C Education Robustness

There has been increasing interest in rising returns to education and rising education costs in the literature. We have so far abstracted from these issues, using the empirical relationship between income and college attainment in 1980 in order to control for changing education rates over time, instead focusing on differential fertility. Is it possible, however, that changes in college returns and costs could be driving changes in differential fertility? In principle, rising education costs could lead to more fertility through a quantity-quality tradeoff, potentially yielding changing patterns of fertility by income. This effect might be mitigated by rising returns to education.

We now allow both the college premium \( \omega \), as described in Equation (1) of the main paper, and education costs \( (p_e) \) to change over time. Relative to our main experiment for 2010, described in Section 4.4 of the main paper, we only need to describe two things: how \( p_e \) changes over time and how \( \omega \) changes.

Children born in 1980 attended college roughly between 1998 and 2002. Thus, we use 2000 as the year to determine costs and returns to college for the 1980 cohort of children. As the 2010 cohort of children has not yet gone to college, we use the 2015 college costs and premium for this cohort. Beginning with \( p_{e,1980} \), we normalize \( p_{e,1980} = 1 \) as before. Although education expenditures map into all possible education-related expenditures per child, we take the stance that college education cost changes accurately describe general changes over time. We therefore choose to proxy the increase in the price of education by the increase in the effective price of college. Using institutional survey data available through the National Center for Education Statistics, we obtain that an annual cost of a public 4-year college is approximately $6,400 in 2000. This includes tuition and room & board, net of grants and scholarships. This quantity for the most recent year available is $7,887, an increase of a 22%. We thus set \( p_{e,2010} = 1.22 \). \( \omega \) in our model
captures the lifetime return to college. This is different from the lifetime college premium which simply refers to the observed difference between the earnings of college graduates and other workers. Thus, we set $\omega_{1980} = 1.29$ and $\omega_{2010} = 1.31$, representing a 29% and 31% college premium, respectively (Valletta Forthcoming).\footnote{Notice that this small increase does not imply that there was a small increase in the returns to education in our time period, simply that the returns to education of children has not changed much. The inequality studied in this paper is focused on the parental generation, that is people who earned wages in 1980 and 2010, rather than people who were born in those years. The increase in inequality for the parental generation was driven at least partially by a large change in returns to college.} We use the same calibration of the model when performing this robustness exercise.\footnote{Note that this involves setting $\hat{\beta} = 2.63$. Recall from Section 3 of the main paper that $\beta = \hat{\beta} \ln(\omega)$. With this parameterization, $\beta$ is the same as in the benchmark.}

The results are quite similar. High income fertility increases by 46% (as opposed to 43.5% in the benchmark model and 40% in the data). In the model, college attainment due to differential fertility rises modestly, by about 2.2 percentage points, but when recalculating holding the cost of marketization constant, this statistic falls by 1.6 percentage points, leading to a total bias from ignoring marketization of about 3.8 percentage points. This result is quantitatively similar to the main exercise.

\[\text{D Normalization of Parameters}\]

\textbf{D.1 Normalizing } p_e

Notice that in our model we can normalize $p_e = 1$ (or any other value), without affecting other meaningful quantities which are mapped to the data. Precisely, we will show that, for any change in $p_e$, parameters of the college attainment function $\pi$ can be adjusted to ensure expenditures $p_n n$, $p_e\bar{e}$ and our targeted moments $\pi$, $n$, $p_m m$, and $t_f$ remain unchanged. At the interior solution we have

\[e = p_n \frac{\beta \theta}{\alpha} - \eta, \text{ or} \]

\[p_e e = p_n \frac{\beta \theta}{\alpha} - p_e \eta.\]
The last equation shows that scaling up \( p_e \) by any factor, say \( \varepsilon \), requires reducing \( e \) and \( \eta \) by the same factor to keep the expenditure on education \( p_e e \) unchanged:

\[
\begin{align*}
p_e e &= p_e \varepsilon \frac{e}{\varepsilon}, \\
p_e \eta &= p_e \varepsilon \frac{\eta}{\varepsilon}.
\end{align*}
\]

It is then seen from Equation (15) of the main paper that the solution to \( n \) will not change due to the scaling above as the product \( \eta p_e \) is unchanged, so is \( p_n \). Note that \( e \) itself has no data counterpart, but the quantity \( \pi (e) \) is used to target college attainment rates in the data. However, the parameters inside \( \pi (\cdot) \) can be scaled as follows, to keep it unchanged:

\[
\pi \left( \frac{e}{\varepsilon} \right) = \ln \left( b \varepsilon^\theta \left( \frac{e}{\varepsilon} + \frac{\eta}{\varepsilon} \right)^\theta \right) = \ln \left( b (e + \eta)^\theta \right) = \pi (e).
\]

Thus, the solution to the model, in terms of \( n \) and \( \pi (e) \), is invariant to the following transformation of parameters:

\[
\tilde{p}_e = p_e \varepsilon, \tilde{\eta} = \frac{\eta}{\varepsilon}, \tilde{b} = b \varepsilon^\theta, \tilde{e} = \frac{e}{\varepsilon} \quad \forall \varepsilon > 0,
\]

The remaining targets are \( p_m m \) and \( t_f \). It is seen from Equations (11) and (12) of the main paper that these quantities remain unchanged as \( n \) remains the same.

**D.2 Normalizing \( p_m \)**

In this section we show that we can normalize \( p_m \) to any value, without affecting any of the meaningful quantities that have a data counterpart: expenditures \( p_n n \), \( p_e e \) and \( n \) and our targeted moments: \( \pi \), \( n \), \( p_m m \), and \( t_f \). The solution to \( p_n \) given in Equation (13) of the main paper:

\[
\begin{align*}
p_n &= \frac{1}{A} \left[ \phi \frac{1}{1 - \rho} \omega_f \frac{p_m}{\rho} + (1 - \phi) \frac{1}{1 - \rho} p_m \frac{p_m}{\rho-1} \right]^{\frac{\rho - 1}{\rho}} \\
&= \left[ A^{\frac{\rho}{1 - \rho}} \phi \frac{1}{1 - \rho} \omega_f \frac{p_m}{\rho} + A^{\frac{\rho}{1 - \rho}} (1 - \phi) \frac{1}{1 - \rho} p_m \frac{p_m}{\rho-1} \right]^{\frac{\rho - 1}{\rho}}.
\end{align*}
\]

\( \Box \)
First we show that when scaling $p_m$ by $\varepsilon > 0$, we can find adjustments to $A$ and $\phi$ to keep $p_n$ unchanged. Let the adjusted parameters be $\tilde{A}$ and $\tilde{\phi}$. We find them by requiring that $p_n$ remains unchanged:

$$
\tilde{A}^{1/p} \phi^{1/p} \frac{1}{w_f^{1/p}} + \tilde{A}^{1/p} (1 - \phi) \frac{1}{w_f^{1/p}} (p_m \varepsilon) \frac{1}{\varepsilon^{1/p}} = A^{1/p} \phi^{1/p} \frac{1}{w_f^{1/p}} + A^{1/p} (1 - \phi) \frac{1}{\varepsilon^{1/p}} p_m^\frac{1}{p}.
$$

For this to hold, $\tilde{A}$ and $\tilde{\phi}$ must satisfy the following two equations:

$$
\tilde{A}^{1/p} \phi^{1/p} = A^{1/p} \phi^{1/p},
$$

$$
\tilde{A}^{1/p} (1 - \phi)^{1/p} = A^{1/p} (1 - \phi)^{1/p}.
$$

Dividing through allows us to find $\tilde{\phi}$.

$$
\frac{\tilde{A}^{1/p} (1 - \phi)^{1/p} \varepsilon^{p-1}}{\tilde{A}^{1/p} \phi^{1/p}} = \frac{A^{1/p} (1 - \phi)^{1/p}}{A^{1/p} \phi^{1/p}},
$$

$$
\left( \frac{1 - \tilde{\phi}}{\phi} \right)^{1/p} = \left( \frac{1 - \phi}{\phi} \right)^{1/p} \varepsilon^{1/p},
$$

$$
1 - \tilde{\phi} = \left( 1 - \frac{\phi}{\phi} \right) \varepsilon^p,
$$

$$
\tilde{\phi} = \frac{1}{1 + \left( 1 - \frac{\phi}{\phi} \right) \varepsilon^p} = \frac{\phi}{\phi + (1 - \phi) \varepsilon^p} \in [0, 1].
$$

Finally, solving for $\tilde{A}$ gives

$$
\tilde{A}^{1/p} = \tilde{A}^{1/p} \left( \frac{\phi}{\phi} \right)^{1/p} = \tilde{A}^{1/p} \left[ \phi + (1 - \phi) \varepsilon^p \right]^{1/p},
$$

$$
\tilde{A} = A \left[ \phi + (1 - \phi) \varepsilon^p \right]^{\frac{1}{p}}.
$$

Thus, scaling $p_m$ by a factor $\varepsilon > 0$, and adjusting the share parameter and productivity as above, keeps $p_n$ fixed. Since $p_n$ is unchanged, the model solution remains the same, and so do the targeted moments for $n$ and $\pi$.

It remains to show that the targeted moments $t_f$ and $p_m m$ also remain unchanged.
We express \( t_f \) and \( m \) in terms of \( p_n \):

\[
(A p_n)^{1-\rho} = \left[ \phi^{1-\rho} w_f^{\rho} + (1 - \phi)^{1-\rho} p_m^{\rho} \right]^{\frac{1}{\rho}}.
\]

Plug the bracketed term into \( t_f \) and \( m \)

\[
t_f = \frac{\left( \frac{1}{w_f} \right)^{1-\rho} \rho}{A \left[ \phi^{1-\rho} w_f^{\rho} + (1 - \phi)^{1-\rho} p_m^{\rho} \right]^{\frac{1}{\rho}}} n = \frac{\left( \frac{1}{w_f} \right)^{1-\rho} \rho}{A (A p_n)^{1-\rho}} n = \frac{\left( \frac{1}{w_f} \right)^{1-\rho} \rho}{A^{1-\rho} \phi^{1-\rho} p_n^{1-\rho}}
\]

\[
m = \frac{\left( \frac{1}{p_m} \right)^{1-\rho} \rho}{A \left[ \phi^{1-\rho} w_f^{\rho} + (1 - \phi)^{1-\rho} p_m^{\rho} \right]^{\frac{1}{\rho}}} n = \frac{\left( \frac{1}{p_m} \right)^{1-\rho} \rho}{A (A p_n)^{1-\rho}} n = \frac{\left( \frac{1}{p_m} \right)^{1-\rho} \rho}{A^{1-\rho} (1 - \phi)^{1-\rho} p_n^{1-\rho}}
\]

We showed that the term \( A^{\frac{\rho}{1-\rho}} \phi^{1-\rho} \) is unchanged due to scaling of \( p_m \), which means that \( t_f \) is unchanged. However, the term \( A^{\frac{\rho}{1-\rho}} (1 - \phi)^{1-\rho} \) increases by a factor of \( \epsilon^{\frac{\rho}{1-\rho}} \). Thus, the effect of scaling \( p_m \) by a factor of \( \epsilon > 0 \), and adjusting \( A \) and \( \phi \) to keep \( p_n \) constant, gives:

\[
mp_m = \frac{\left( \frac{1}{p_m} \right)^{1-\rho} \rho (p_m \epsilon)}{A^{\frac{\rho}{1-\rho}} (1 - \phi)^{1-\rho} \epsilon^{\frac{\rho}{1-\rho}} p_n^{1-\rho}} = \frac{p_m^{\rho} \epsilon^{\frac{\rho}{1-\rho}}}{A^{\frac{\rho}{1-\rho}} (1 - \phi)^{1-\rho} \epsilon^{\frac{\rho}{1-\rho}} p_n^{1-\rho}} = \frac{\frac{\epsilon^{\frac{\rho}{1-\rho}}}{p_m^{\rho}}}{A^{\frac{\rho}{1-\rho}} (1 - \phi)^{1-\rho} \epsilon^{\frac{\rho}{1-\rho}} p_n^{1-\rho}}
\]

Notice that \( \epsilon \) cancels out, and therefore does not affect \( mp_m \).