Search Model of Unemployment

1 Introduction

The Neoclassical Growth Model is not suitable for the study of one of the most important macroeconomic variables - unemployment. In particular, the NGM assumes full employment all the time. In the real world however, we never observe zero unemployment rates, not even during economic booms. Labor economists built models that address this issue, and perhaps the most important contribution is the search model of unemployment. The main distinguishing features of search models are: (i) individuals are heterogeneous (some are employed while others not), and (ii) it takes time to transit from unemployment to employment. The search model we will present in these notes is based on McCall (1970) and Phelps (1970).

2 Basic Model of Job Search

Suppose that a single individual wants to maximize his expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(w_t)$$

In each period the worker can be in one of two states: (i) employed and (ii) unemployed. We only analyze the decision of unemployed person. In each period of unemployment, the worker gets a job offer \(w_t\), and he needs to choose whether to accept it or not. Suppose that at time \(t\) an unemployed individual accepts a job offer \(w_t\). Then he becomes a worker and enjoys the wage \(w_t\) from that time on. If the unemployed individual rejects the offer \(w_t\) then he gets unemployment insurance benefit \(b_t\) and resumes the search in the next period. Suppose that the wage offers are drawn in each period from some distribution with probability density \(f(\cdot)\) and cumulative density \(F(\cdot)\). That is \(F(x) = \Pr(w \leq x) = \int_{-\infty}^{x} f(s) \, ds\).

The only state variable in this model is \(w_t\) and we can define the function \(V(w_t)\) that gives the maximal possible continuation utility that can be attained with offer \(w_t\):

$$V(w_t) = \max \left\{ \frac{u(w_t)}{1-\beta}, u(b_t) + \beta E[V(w_{t+1})] \right\}$$

The first term in the curly brackets is the utility attained after accepting an offer \(w_t\):

$$\sum_{s=t}^{\infty} \beta^{s-t} u(w_t) = \frac{u(w_t)}{1-\beta}$$

The second term in the curly brackets gives the expected value that can be attained when the individual rejects the offer and resumes search at time \(t + 1\). This value is constant which is independent of the current offer, and we denote it with \(\kappa \equiv E[V(w_{t+1})]\). The household’s choice between accepting a given job offer or not can be illustrated in the next diagram.
Notice that the first term in the curly brackets of (2) is the utility from accepting an offer \( w \), and is represented by an increasing function in the above figure. The second term in the curly brackets of (2) is the utility from rejecting an offer \( w \), and is independent of the offer being rejected. This term is represented by the horizontal line in the above figure.

Notice that an unemployed person will accept wage offers that are at least \( w^* \), and this threshold wage is called reservation wage. Formally, the reservation wage is the minimal wage offer that is acceptable by unemployed, and is determined by the following equation:

\[
\frac{u(w^*)}{1-\beta} = u(b) + \beta \kappa
\]  

Equation (3) can be solved for the reservation wage, provided that the period utility function \( u \) is given and we have computed \( \kappa = E[V(w_{t+1})] \), the expected utility from rejecting an offer and renewing the search in the next period. Next, we do exactly that.

\[
\kappa = E[V(w_{t+1})] = \int_{-\infty}^{w^*} V(w_{t+1}) \, dF(w_{t+1}) + \int_{w^*}^{\infty} V(w_{t+1}) \, dF(w_{t+1})
\]  

Notice that we split the domain of integration into two parts, realizing that in the next period an offer can still be rejected (if \( w_{t+1} < w^* \)) or accepted (if \( w_{t+1} \geq w^* \)). If the offer is rejected, the continuation utility will be \( u(b) + \beta \kappa \). Accepted offers can have any value that is above \( w^* \), so we need to average (integrate) over all the possible wages that are accepted. The second term in the last equation is \( E[w_{t+1} \mid w_{t+1} \geq w^*] = \int_{w^*}^{\infty} V(w_{t+1}) \, dF(w_{t+1}) \), i.e. the conditional expectation given that the offer exceeds the reservation wage. Simplifying
further leads to:

\[ \kappa = \int_{-\infty}^{w^*} [u(b) + \beta \kappa] dF(w_{t+1}) + \int_{w^*}^{\infty} \frac{u(w_{t+1})}{1 - \beta} dF(w_{t+1}) \]

\[ = [u(b) + \beta \kappa] \int_{-\infty}^{w^*} dF(w_{t+1}) + \int_{w^*}^{\infty} \frac{u(w_{t+1})}{1 - \beta} dF(w_{t+1}) \]

\[ = [u(b) + \beta \kappa] \int_{-\infty}^{w^*} f(w_{t+1}) dw_{t+1} + \int_{w^*}^{\infty} \frac{u(w_{t+1})}{1 - \beta} dF(w_{t+1}) \]

\[ = F(w^*) [u(b) + \beta \kappa] + \int_{w^*}^{\infty} \frac{u(w_{t+1})}{1 - \beta} dF(w_{t+1}) \]

Substituting from equation (3) gives:

\[ \kappa (w^*) = F(w^*) \frac{u(w^*)}{1 - \beta} + \int_{w^*}^{\infty} \frac{u(w_{t+1})}{1 - \beta} dF(w_{t+1}) \]  

(4)

**Example 1** Find the reservation wage when the distribution of wage offer is uniform on the interval \([0, \bar{w}]\). For simplicity assume that \(u(w) = w\).

We plug equation (4) into (3) and get one equation with one unknown \(w^*\).

\[ \frac{w^*}{1 - \beta} = b + \beta \left[ F(w^*) \frac{w^*}{1 - \beta} + \int_{w^*}^{\infty} \frac{x}{1 - \beta} dF(x) \right] \]

Recall that the uniform density is \(f(x) = \frac{1}{\bar{w}}\) and the cumulative density is \(F(x) = \frac{x}{\bar{w}}\). The integral in the last equation is

\[ \int_{w^*}^{\bar{w}} \frac{x}{1 - \beta} \frac{1}{\bar{w}} dx = \frac{1}{(1 - \beta) \bar{w}} \int_{w^*}^{\bar{w}} x dx = \frac{\bar{w}^2 - w^*^2}{2 (1 - \beta) \bar{w}} \]

Thus,

\[ \frac{w^*}{1 - \beta} = b + \beta \left[ \frac{w^*}{\bar{w}} \frac{w^*}{1 - \beta} + \frac{\bar{w}^2 - w^*^2}{2 (1 - \beta) \bar{w}} \right] \]

\[ w^* = b (1 - \beta) + \beta \left[ \frac{w^*^2}{\bar{w}} + \frac{\bar{w}^2 - w^*^2}{2 \bar{w}} \right] \]

\[ 2\bar{w} w^* = 2\bar{w} b (1 - \beta) + \beta w^*^2 + \beta \bar{w}^2 \]

\[ \beta w^*^2 - 2\bar{w} w^* + 2\bar{w} b (1 - \beta) + \beta \bar{w}^2 = 0 \]

\[ w^* = \frac{2\bar{w} \pm \sqrt{4\bar{w}^2 - 4 \cdot 2\bar{w} b (1 - \beta) + 4\beta^2 \bar{w}^2}}{2\beta} \]

\[ w^* = \frac{2\bar{w} \pm 2\sqrt{\bar{w}^2 - 2\bar{w} b (1 - \beta) + \beta^2 \bar{w}^2}}{2\beta} \]

\[ w^* = \frac{\bar{w} \pm \sqrt{\bar{w}^2 (1 - \beta^2) - 2\bar{w} b (1 - \beta)}}{\beta} \]
The above quadratic equation has two solutions, in one of them the reservation wage is greater than the maximal possible wage, even when there is no unemployment insurance benefits \((b = 0)\). This cannot be valid solution because the household will never accept any offers. Therefore, the only admissible solution is:

\[
w^* = \frac{\bar{w} - \sqrt{\bar{w}^2 (1 - \beta^2) - 2\bar{w}b (1 - \beta)}}{\beta}
\]

Notice that the reservation wage is increasing \(b\), so higher unemployment insurance benefit makes this household more picky and more reluctant to accept an offer. As an exercise, show that there is high enough \(b\) that this household will not accept any offers \((w^* \geq \bar{w})\).

As another exercise, show that \(w^*\) is also increasing in \(\beta\). Recall that \(\beta\) is the time discount factor. Higher \(\beta\) means that this household is more patient and puts more weight on the future. Therefore, it is intuitive that the household will set higher reservation wage and as a result will spend more time searching.

**Example 2** Find the distribution of observed wages for the general case and for the uniform distribution in previous example.

Let the cumulative density of observed wages be \(G(x)\). This function gives the conditional probability of offer being less than \(x\), given that the wage is observed, i.e. no less than the reservation wage. Thus,

\[
G(x) = \frac{\Pr(w \leq x | w \geq w^*)}{\Pr(w \geq w^*)} = \frac{\int_{w^*}^{x} f(x) \, dx}{1 - F(w^*)} = \frac{F(x) - F(w^*)}{1 - F(w^*)}
\]

Obviously, the cumulative density cannot have negative values, so for \(x \leq w^*\) we have \(G(x) = 0\). The cumulative density can therefore be written as

\[
G(x) = \max \left\{ \frac{F(x) - F(w^*)}{1 - F(w^*)}, 0 \right\}
\]

For the uniform distribution of wage offers we have

\[
G(x) = \max \left\{ \frac{x - w^*}{\bar{w} - w^*}, 0 \right\} = \max \left\{ \frac{x - w^*}{\bar{w} - w^*}, 0 \right\} = \max \left\{ \frac{x - w^*}{\bar{w} - w^*}, 0 \right\}
\]

Thus, the distribution of observed wages is also uniform, on the interval \([0, \bar{w} - w^*]\). To check that the answer is correct, verify that \(G(\bar{w}) = 1\).
References
