Business Cycles part 1

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Business Cycles: Introduction
Main message from Solow and NGM: without growth in productivity, it is impossible to achieve sustained growth in standard of living.

Conclusion: need theory of productivity, in order to better understand growth.

Q. What if business cycles are also driven by productivity?
Introduction

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Q. What if business cycles are also driven by productivity?
Finn E. Kydland & Edward C. Prescott: Nobel Memorial Prize in Economics in 2004

"for their contributions to dynamic macroeconomics: the time consistency of economic policy and the driving forces behind business cycles."
Introduction

**Real gross domestic product per capita**

- Pre recession growth trend: 2.1% annual
- Business cycles
- Post recession growth: 1.58%
- Growth since 2017: 1.8%

Source: US. Bureau of Economic Analysis/FRED
Economic growth and business cycles are separate phenomena, but they both driven by productivity:

- Economic growth is driven by growth in productivity - \( A_t = A_0 (1 + \gamma_A)^t \).
- Business cycles are driven by stochastic shocks to productivity (RBC - Real Business Cycle theory).

\[
A_t = A_0 (1 + \gamma_A)^t \times e^{zt}
\]

where \( z_t = \rho z_{t-1} + \varepsilon_t \) and \( \varepsilon_t \sim i.i.d. N \left( 0, \sigma^2 \varepsilon \right) \)

Q. Why did we choose this particular stochastic process for productivity shocks?
A. Goal: replicating observed business cycles in the data.
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Kydland and Prescott’s Theory

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Stochastic NGM
Social Planner’s Problem:

\[
\begin{aligned}
\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} & \quad E_0 \sum_{t=0}^{\infty} \beta^t u (c_t, 1 - h_t) \\
\text{s.t.} & \\
[\text{Feasibility}] & \quad c_t + k_{t+1} = A_t k_t^\theta h_t^{1-\theta} + (1 - \delta) k_t \quad \forall t \\
[\text{Productivity}] & \quad A_t = A_0 (1 + \gamma_A)^t e^{z_t}, \quad z_t = \rho z_{t-1} + \varepsilon_t \\
& \quad \text{and } \varepsilon_t \sim i.i.d. \ N (0, \sigma^2_\varepsilon) \\
& \quad k_0 > 0 \text{ given}
\end{aligned}
\]
Stochastic NGM

- $E_0$ is shorthand for Conditional Expectation. In general

$$E_t(X) = E(X_t|\Omega_t)$$

where $\Omega_t$ is the information available at time $t$. Here the information known at time $t$ is

$$\Omega_t = (A_t, k_t)$$

- Remind yourself what is conditional expectation by example. Let $X$ be a toss of a die.
  - $E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6 = 3.5$
  - $E(X|X \text{ is even}) = \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 4 + \frac{1}{3} \cdot 6 = 4$
  - $E(X|X \text{ is odd}) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 5 = 3$
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**Stochastic NGM**

- $e^{zt}$ is the stochastic portion of productivity
  $A_t = A_0 \left(1 + \gamma_A \right)^t e^{zt}$

- $zt = \rho z_{t-1} + \varepsilon_t$, with $\varepsilon_t \sim i.i.d. \ N(0, \sigma_\varepsilon^2)$

  is called Autoregressive stochastic process of order 1 (short: AR(1)).

- The parameter $\rho$ is is the autoregressive coefficient. We assume $|\rho| < 1$.

- $\varepsilon_t$ is called white noise process (aka *innovation* process).

- *i.i.d.* stands for independent and identically distributed random variables. Thus, all $\varepsilon_t$ have the same mean 0 and same variance $\sigma_\varepsilon^2$, and $\varepsilon_t$ and $\varepsilon_{t+k}$ are independent for all $k = \pm 1, 2, ...
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Stochastic NGM

- $e^{z_t}$ is the stochastic portion of productivity
  \[ A_t = A_0 \left( 1 + \gamma_A \right)^t e^{z_t} \]

\[ z_t = \rho z_{t-1} + \varepsilon_t, \text{ with } \varepsilon_t \sim i.i.d. \ N \left( 0, \sigma^2_\varepsilon \right) \]

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Stochastic NGM: Recap

Social Planner’s Problem:

$$\max \{c_t, h_t, k_{t+1}\}_{t=0}^\infty E_0 \sum_{t=0}^\infty \beta^t u(c_t, 1 - h_t)$$

s.t.

[Feasibility] : $$c_t + k_{t+1} = A_t k_t^\theta h_t^{1-\theta} + (1 - \delta) k_t \ \forall t$$

[Productivity] : $$A_t = A_0 (1 + \gamma_A)^t e^{zt}, \ z_t = \rho z_{t-1} + \varepsilon_t$$

: and $$\varepsilon_t \sim i.i.d. N(0, \sigma^2_{\varepsilon})$$

$$k_0 > 0 \ \text{given}$$
Stochastic NGM: Equilibrium Conditions

The necessary conditions for optimal \( \{c_t, h_t, k_{t+1}\}_{t=0}^\infty \) are:

\[
\begin{align*}
\text{[Labor]} & : \quad \frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = (1 - \theta) A_t k_t^\theta h_t^{-\theta} \\
\text{[EE]} & : \quad u_1(c_t, 1-h_t) \\
& = \beta E_t \left\{ u_1(c_{t+1}, 1-h_{t+1}) \left[ \theta A_{t+1} k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + 1 - \delta \right] \right\} \\
\text{[Feas]} & : \quad c_t + k_{t+1} = A_t k_t^\theta h_t^{1-\theta} + (1 - \delta) k_t
\end{align*}
\]
Productivity Process
We discuss the properties of $z_t$ in

$$A_t = A_0 \left(1 + \gamma_A\right)^t e^{z_t}$$

where

$$z_t = \rho z_{t-1} + \epsilon_t \text{ and } \epsilon_t \sim i.i.d. \ N(0, \sigma_{\epsilon}^2)$$

$$|\rho| < 1$$

The point of choosing this particular process: replicating observed business cycles in the data.
AR(1) Process

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$$|\rho| < 1$$

- The point of choosing this particular process: replicating observed business cycles in the data.
AR(p) Process

\[
\begin{align*}
[AR (1)] & : z_t = \rho z_{t-1} + \varepsilon_t \\
[AR (2)] & : z_t = \rho_1 z_{t-1} + \rho_2 z_{t-2} + \varepsilon_t \\
& \vdots \\
[AR (p)] & : z_t = \rho_1 z_{t-1} + \rho_2 z_{t-2} + \ldots + \rho_p z_{t-p} + \varepsilon_t
\end{align*}
\]
By recursive substitution

\[ z_t = \rho (\rho z_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \]

\[ = \rho^2 z_{t-2} + \rho \varepsilon_{t-1} + \varepsilon_t \]

\[ \vdots \]

\[ = \lim_{k \to \infty} \rho^k z_{t-k} + \sum_{k=0}^{\infty} \rho^k \varepsilon_{t-k} \]

Thus the \( MA(\infty) \) representation of \( AR(1) \) is:

\[ z_t = \varepsilon_t + \rho \varepsilon_{t-1} + \rho^2 \varepsilon_{t-2} + \rho^3 \varepsilon_{t-3} + \ldots \]
AR(1) Process: Mean

- **Theorem:** $E(z_t) = 0 \ \forall t$.
- **Proof.** MA($\infty$) representation of $z_t$:

$$z_t = \varepsilon_t + \rho \varepsilon_{t-1} + \rho^2 \varepsilon_{t-2} + ...$$

Taking expectation:

$$E(z_t) = \underbrace{E(\varepsilon_t)}_{=0} + \underbrace{\rho E(\varepsilon_{t-1})}_{=0} + \underbrace{\rho^2 E(\varepsilon_{t-2})}_{=0} + ... = 0$$
Theorem: \( E (z_t) = 0 \ \forall t. \)

Proof. \( MA(\infty) \) representation of \( z_t \):

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z_t = \epsilon_t + \rho \epsilon_{t-1} + \rho^2 \epsilon_{t-2} + ... \]

Taking expectation:

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E (z_t) = E (\epsilon_t) + \rho E (\epsilon_{t-1}) + \rho^2 E (\epsilon_{t-2}) + ... = 0
\]
**AR(1) Process: Variance**

- **Theorem:** \( \text{Var} \left( z_t \right) = \frac{\sigma^2_\varepsilon}{1-\rho^2} \forall t. \)

- **Proof.** \( MA(\infty) \) reprezentation of \( z_t \):

\[
z_t = \varepsilon_t + \rho \varepsilon_{t-1} + \rho^2 \varepsilon_{t-2} + ... \]

Thus

\[
\text{Var} \left( z_t \right) = \text{Var} \left( \varepsilon_t \right) + \rho^2 \text{Var} \left( \varepsilon_{t-1} \right) + \rho^4 \text{Var} \left( \varepsilon_{t-2} \right) + ... \\
= \sigma^2_\varepsilon + \rho^2 \sigma^2_\varepsilon + \rho^4 \sigma^2_\varepsilon + ... \\
= \sigma^2_\varepsilon \sum_{t=0}^{\infty} (\rho^2)^t = \frac{\sigma^2_\varepsilon}{1-\rho^2} \]
AR(1) Process: Variance

- **Theorem:** \( \text{Var} (z_t) = \frac{\sigma_{\varepsilon}^2}{1 - \rho^2} \forall t. \)

- **Proof.** MA(\( \infty \)) representation of \( z_t \):

\[
z_t = \varepsilon_t + \rho \varepsilon_{t-1} + \rho^2 \varepsilon_{t-2} + \ldots
\]

Thus

\[
\text{Var} (z_t) = \text{Var} (\varepsilon_t) + \rho^2 \text{Var} (\varepsilon_{t-1}) + \rho^4 \text{Var} (\varepsilon_{t-2}) + \ldots
\]

\[
= \sigma_{\varepsilon}^2 + \rho^2 \sigma_{\varepsilon}^2 + \rho^4 \sigma_{\varepsilon}^2 + \ldots
\]

\[
= \sigma_{\varepsilon}^2 \sum_{t=0}^{\infty} (\rho^2)^t = \frac{\sigma_{\varepsilon}^2}{1 - \rho^2}
\]
Theorem: $\text{Cov}(z_t, z_{t-k}) = \rho^k \left( \frac{\sigma^2}{1-\rho^2} \right) \forall t.$

Proof. Start with $k = 1.$

\[
\text{Cov}(z_t, z_{t-1}) = \text{Cov}(\rho z_{t-1} + \epsilon_t, z_{t-1}) \\
= \rho \text{Cov}(z_{t-1}, z_{t-1}) + \text{Cov}(\epsilon_t, z_{t-1}) \\
= \rho \text{Var}(z_t) + 0
\]
Theorem: $\text{Cov} (z_t, z_{t-k}) = \rho^k \left( \frac{\sigma^2 \epsilon}{1-\rho^2} \right) \forall t.$

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= \rho \text{Var} (z_t) + 0
\]
**AR(1) Process: Covariance**

- **Proof.** next $k = 2$.

\[
\text{Cov} \left( z_t, z_{t-2} \right) = \text{Cov} \left( \rho z_{t-1} + \epsilon_t, z_{t-2} \right) \\
= \rho \text{Cov} \left( z_{t-1}, z_{t-2} \right) \\
= \rho^2 \text{Var} \left( z_t \right)
\]

- Thus

\[
\text{Cov} \left( z_t, z_{t-k} \right) = \rho^k \text{Var} \left( z_t \right) = \rho^k \left( \frac{\sigma^2 \epsilon}{1 - \rho^2} \right)
\]
AR(1) Process: Covariance

Proof. next \( k = 2. \)

\[
Cov \left( z_t, z_{t-2} \right) = Cov \left( \rho z_{t-1} + \epsilon_t, z_{t-2} \right) \\
= \rho Cov \left( z_{t-1}, z_{t-2} \right) \\
= \rho^2 Var \left( z_t \right)
\]

Thus

\[
Cov \left( z_t, z_{t-k} \right) = \rho^k Var \left( z_t \right) = \rho^k \left( \frac{\sigma_\epsilon^2}{1 - \rho^2} \right)
\]
AR(1) Process: Correlation

\[
Corr(z_t, z_{t-k}) = \frac{\text{Cov}(z_t, z_{t-k})}{\sqrt{\text{Var}(z_t)} \sqrt{\text{Var}(z_{t-k})}} = \frac{\rho^k \text{Var}(z_t)}{\sqrt{\text{Var}(z_t)} \sqrt{\text{Var}(z_t)}} = \rho^k
\]
AR(1) Process: Prediction

- One period ahead:
  \[
  E (z_{t+1}|z_t) = E (\rho z_t + \varepsilon_{t+1}|z_t) \\
  = E (\rho z_t|z_t) + E (\varepsilon_{t+1}|z_t) \\
  = \rho z_t + 0
  \]

- Two periods ahead:
  \[
  E (z_{t+2}|z_t) = E (\rho z_{t+1} + \varepsilon_{t+2}|z_t) \\
  = \rho E (z_{t+1}|z_t) \\
  = \rho^2 z_t
  \]

- \( k \) periods ahead
  \[
  E (z_{t+k}|z_t) = \rho^k z_t
  \]
AR(1) Process: Prediction

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  \[
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  \]

- \(k\) periods ahead
  \[
  E(z_{t+k}|z_t) = \rho^k z_t
  \]
AR(1) Process: Prediction

- Example. Suppose that $\rho = 0.9$ and the shock today is $z_t = -1$ (recession).

- Predicted shock next period:

$$ E (z_{t+1} | z_t) = \rho^1 z_t = 0.9^1 \cdot -1 = -0.9 $$

- Predicted shock 10 periods ahead:

$$ E (z_{t+10} | z_t) = \rho^{10} z_t = 0.9^{10} \cdot -1 = -0.34868 $$

- Predicted shock 40 periods ahead:

$$ E (z_{t+40} | z_t) = \rho^{40} z_t = 0.9^{40} \cdot -1 = -0.0148 $$
Example. Suppose that $\rho = 0.9$ and the shock today is $z_t = -1$ (recession).

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$$E(z_{t+1}|z_t) = \rho^1 z_t = 0.9^1 \cdot -1 = -0.9$$

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Predicted shock 40 periods ahead:

$$E(z_{t+40}|z_t) = \rho^{40} z_t = 0.9^{40} \cdot -1 = -0.0148$$
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Predicted shock 10 periods ahead:

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Predicted shock 40 periods ahead:

$$E (z_{t+40} | z_t) = \rho^{40} z_t = 0.9^{40} \cdot -1 = -0.0148$$
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- Example. Suppose that $\rho = 0.9$ and the shock today is $z_t = -1$ (recession).
- Predicted shock next period:
  \[ E(z_{t+1} \mid z_t) = \rho^1 z_t = 0.9^1 \cdot -1 = -0.9 \]
- Predicted shock 10 periods ahead:
  \[ E(z_{t+10} \mid z_t) = \rho^{10} z_t = 0.9^{10} \cdot -1 = -0.34868 \]
- Predicted shock 40 periods ahead:
  \[ E(z_{t+40} \mid z_t) = \rho^{40} z_t = 0.9^{40} \cdot -1 = -0.0148 \]
AR(1) Process: Prediction

- $\rho = -0.9$
- $\rho = 0$
- $\rho = 0.9$
- $\rho = 1$
AR(1) Process: Summary of Properties

[Mean] : \( E (z_t) = 0 \)

[Variance] : \( Var (z_t) = \frac{\sigma^2_\varepsilon}{1 - \rho^2} \)

[Autocovariance] : \( Cov (z_t, z_{t-k}) = \rho^k \left( \frac{\sigma^2_\varepsilon}{1 - \rho^2} \right) \)

[Autocorrelation] : \( Corr (z_t, z_{t-k}) = \rho^k \)

[Prediction] : \( E (z_{t+k} \mid z_t) = \rho^k z_t \)
Estimating the Productivity Parameters
Recall

\[ Y_t = A_t k_t^\theta h_t^{1-\theta} \]

where \( A_t = A_0 \left(1 + \gamma_A\right)^t e^{zt}, \]
\[ z_t = \rho z_{t-1} + \epsilon_t, \]
\[ \epsilon_t \sim i.i.d. \ N\left(0, \sigma^2_\epsilon\right) \]

Goal: estimate \( \gamma_A, \rho, \sigma^2_\epsilon \).
Estimating the AR(1) process

- Recall

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Y_t = A_t k_t^\theta h_t^{1-\theta}
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where \( A_t = A_0 (1 + \gamma_A)^t e^{z_t}, \) \( z_t = \rho z_{t-1} + \varepsilon_t, \)

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Under the assumption of Cobb-Douglas production function, we have:

\[ Y_t = A_t k_t^\theta h_t^{1-\theta} \]

\[ \Rightarrow A_t = \frac{Y_t}{k_t^\theta h_t^{1-\theta}} \]

Suppose that we previously calibrated \( \theta = 0.35 \) (capital share), we need data on:

- \( Y_t \) - real GDP
- \( k_t \) - real stock of fixed assets (capital in NIPA)
- \( h_t \) - total number of hours worked in the economy
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Estimating \( \gamma_A \) and \( z_t \)

Having data on \( A_t \) and using the form of \( A_t \),

\[
A_t = A_0 (1 + \gamma_A)^t e^{u_t}
\]

Taking log

\[
\log (A_t) = \log (A_0) + t \log (1 + \gamma_A) + u_t
\]

Estimate (OLS)

\[
\log (A_t) = \beta_0 + \beta_1 \cdot t + u_t
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\( \hat{\beta}_1 = \log (1 + \gamma_A) \), \( \Rightarrow \hat{\gamma}_A = \exp(\hat{\beta}_1) - 1 \).

\( z_t = \log (A_t) - (\hat{\beta}_0 + \hat{\beta}_1 \cdot t) \), residuals (estimated \( u_t \)).
Estimating $\gamma_A$ and $z_t$

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Estimating $\rho$

- Having times series on $z_t$, use the AR(1) model:

\[ z_t = \rho z_{t-1} + \epsilon_t \]

- Option 1: estimate $\rho$ with OLS.
- Option 2: estimate $\rho$ as $\text{Corr} (z_t, z_{t-1})$.

- Both methods are not efficient, but consistent.
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Both methods are not efficient, but consistent.
Estimating $\sigma^2_\varepsilon$

Having obtained $\hat{\rho}$, we have

$$z_t = \hat{\rho}z_{t-1} + \hat{\epsilon}_t$$

- Option 1:
  $$\hat{\sigma}^2_\varepsilon = \frac{1}{n-1} \sum_{t=1}^{n} (z_t - \hat{\rho}z_{t-1})^2$$

- Option 2:
  $$\text{Var}(z_t) = \frac{1}{n-1} \sum_{t=1}^{n} (z_t - \bar{z}_t)^2$$
  $$\hat{\sigma}^2_\varepsilon = \text{Var}(z_t) \left(1 - \hat{\rho}^2\right)$$
- Estimating $\sigma^2_\varepsilon$
- Having obtained $\hat{\rho}$, we have

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### Matlab code

<table>
<thead>
<tr>
<th>Step</th>
<th>Matlab</th>
</tr>
</thead>
</table>
| \( \log(A_t) = \beta_0 + \beta_1 \cdot t + u_t \) | \( X = [\text{ones(length}(t),1), t] \)  
\( b = X \backslash \log(A); \) |
| Residuals                                 | \( z = \log(A) - X \backslash b; \)                                  |
| Define \( z_t \)                          | \( z_t = z(2:\text{end}); \)                                         |
| Define \( z_{t-1} \)                      | \( z_{\text{lag}} = z(1:\text{end}-1); \)                           |
| \( z_t = \rho z_{t-1} + \epsilon_t \)    | \( \text{rho} = z_{\text{lag}} \backslash z_t; \)                   |
| estimate \( \sigma_\epsilon \)           | \( \text{sigma}_e = \text{std}(z_t - \rho \cdot z_{\text{lag}}); \) |