Cash In Advance Model

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May 7, 2020
Questions in Monetary Economics

1. What causes inflation?
2. What determines the amount of seigniorage?
3. What are the real effects of increasing money supply?
4. What is the optimal monetary policy?
Cash In Advance (CIA) model

Model

- **Money supply:**
  
  $$m_{t+1}^s = (1 + \mu_{t+1}) m_t^s$$
  
  where
  $$\mu_{t+1} m_t^s = \tau_t$$

- **Bonds:** $b_t$, pay interest rate $i_t$.

- **Cash in advance constraint:** $p_t c_t \leq m_t^d$. In equilibrium, must have
  
  $$\frac{m_t^d}{p_t} = c_t$$
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Model

Household problem:

\[
\max \left\{ c_t, h_t, b^d_{t+1}, m^d_{t+1} \right\}_{t=0}^{\infty} \sum_{t=0}^{\infty} \beta^t u (c_t, 1 - h_t)
\]

[B.C.]: \( p_t c_t + m^d_{t+1} + b_{t+1} = m^d_t + (1 + i_t) b_t + p_t h_t + \tau_t \quad \forall t \)

[CIA]: \( p_t c_t = m^d_t \quad \forall t \)

Production: \( y_t = h_t \quad \Rightarrow \quad w_t = p_t \).
Equilibrium (market clearing) conditions

- [Labor] : \( y_t = h_t \)
- [Goods] : \( y_t = c_t \)
- [Money] : \( m^s_t = m^d_t = m_t \)
- [Bonds] : \( b^s_t = b^d_t = 0 \)
Results 1: General
Equilibrium conditions

- Equilibrium conditions. Given exogenous sequence of money supply $\{m_t\}_{t=0}^{\infty}$, the equilibrium time paths of $\{c_t, p_t, i_t\}_{t=0}^{\infty}$ must solve for all $t = 0, 1, 2, ...$

\[
[h_t] : \quad \frac{u_2(c_t, 1 - c_t)}{u_1(c_t, 1 - c_t)} = \frac{1}{1 + i_t}
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[EE] : \quad \frac{u_1(c_t, 1 - c_t)}{u_1(c_{t+1}, 1 - c_{t+1})} = \beta \frac{p_t (1 + i_t)}{p_{t+1}}
\]
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[CIA] : \quad p_t c_t = m_t
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Intuition: optimal labor condition

The optimal labor condition is:

\[
\frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = \frac{p_t}{p_t (1 + i_t)}
\]

In NGM, this condition gives \( MRS_{l,c} = \frac{P_l}{P_c} \). Here price of leisure (wage) = price of consumption = \( p_t \). So why we don’t have

\[
\frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} = \frac{p_t}{p_t} = 1 \quad ?
\]

- CIA constraint distorts the above condition. If the household wants to buy extra units of consumption at time \( t \), it needs to bring \( p_t \) extra units of money \( m_t \) (reduce bonds \( b_t \) by that amount) and give up interest. Thus, the cost of current period consumption becomes \( p_{c_t} = p_t (1 + i_t) \) because of the CIA constraint.
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- Observe that if \( i_t = 0 \), then we get the same non-distorted (Pareto Optimal) condition as in NGM. The condition of \( i_t = 0 \) is called the Friedman Rule - a condition for optimal monetary policy.

- Also observe that under Friedman Rule, \( i_t = 0 \Rightarrow \pi \approx -r \).
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- Also observe that under Friedman Rule, \( i_t = 0 \Rightarrow \pi \approx -r \).
Intuition: Euler Equation

The inter-temporal optimality condition is:

\[ u_1(c_t, 1 - c_t) = \beta u_1(c_{t+1}, 1 - c_{t+1}) \frac{p_t (1 + i_t)}{p_{t+1}} \]

Reminds the NGMs "pain" = "gain" condition.

- The LHS is the "pain" from giving up one unit of \( c_t \).
- This allows reducing the beginning of period money, \( m_t \), by \( p_t \) and buy additional bonds \( b_t \) for that amount. These bonds in turn allow increasing \( m_{t+1} \) by \( p_t (1 + i_t) \) and in period \( t + 1 \) we can buy \( p_t (1 + i_t) / p_{t+1} \) units of \( c_{t+1} \). Thus, the RHS is the present value of the utility gain from giving up 1 unit of \( c_t \) and investing in bonds.
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It is possible to show that if CIA constraint is not binding, or if Friedman Rule is implemented, then the Euler Equation becomes:

\[
\begin{align*}
    u_1 (c_t, 1 - c_t) &= \beta u_1 (c_{t+1}, 1 - c_{t+1}) \frac{p_t (1 + i_{t+1})}{p_{t+1}} \\
    u_1 (c_t, 1 - c_t) &= \beta u_1 (c_{t+1}, 1 - c_{t+1}) (1 + r_{t+1})
\end{align*}
\]

Which is the same as in NGM (Pareto Optimal):

\[1 + r_{t+1} = 1 + F_1 (k_{t+1}, h_{t+1}) - \delta.\]

Q. Prove that if CIA constraint is not binding, then \(i_t = 0\) (Friedman Rule). Hint: use the first order condition with \(\phi_t = 0\ \forall t\).
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$$u_1(c_t, 1 - c_t) = \beta u_1(c_{t+1}, 1 - c_{t+1})(1 + r_{t+1})$$

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Q. Prove that if CIA constraint is not binding, then $i_t = 0$ (Friedman Rule). Hint: use the first order condition with $\phi_t = 0 \forall t$. 
Consider constant money growth policy: \( m_{t+1} = (1 + \mu) m_t \).

**Results:**

1. \( \pi = \mu \) (inflation rate = rate of growth of money supply).
2. \( r = \rho \) (real interest rate = time discount rate in \( \beta = \frac{1}{1+\rho} \)).
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**Results:**

1. $\pi = \mu$ (inflation rate = rate of growth of money supply).
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Results2: Cobb-Douglas Utility
Equilibrium conditions.

Given exogenous sequence of money supply \( \{ m_t \}_{t=0}^{\infty} \), the equilibrium time paths of \( \{ c_t, p_t, i_t \}_{t=0}^{\infty} \) must solve for all \( t = 0, 1, 2, ... \)

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[h_t] : \frac{u_2 (c_t, 1 - c_t)}{u_1 (c_t, 1 - c_t)} = \frac{1}{1 + i_t}
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\[
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With Cobb-Douglas utility become

\[
[h_t] : \frac{1 - \alpha}{\alpha} \frac{c_t}{1 - c_t} = \frac{1}{1 + i_t}
\]

\[
[EE] : \frac{c_{t+1}}{c_t} = \beta \frac{p_t (1 + i_t)}{p_{t+1}}
\]

\[
[CIA] : p_t c_t = m_t
\]
Cobb-Douglas utility: analytical solution

\[ i_t = \frac{1 + \mu_{t+1}}{\beta} - 1 \]

\[ c_t = y_t = h_t = \frac{1}{1 + \frac{1-\alpha}{\alpha} (1 + i_t)} = \frac{\beta}{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_{t+1})} \]

\[ p_t = \frac{m_t}{\beta} \left( \beta + \frac{1-\alpha}{\alpha} (1 + \mu_{t+1}) \right) \]

\[ \pi_t = (1 + \mu_t) \frac{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_{t+1})}{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_t)} - 1 \]

\[ r_t = \frac{1}{\beta} \left[ \frac{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_t)}{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_{t+1})} \right] - 1 \]
Cobb-Douglas utility: analytical solution

\[ i_t = \frac{1 + \mu_{t+1}}{\beta} - 1 \]

- Nominal interest rate is increasing in the rate of future money growth.
- Friedman Rule, \( i_t = 0 \), implies that \( \mu_{t+1}^* \approx -\rho \). That is, optimal monetary policy in this model is to decrease the money supply at the rate of time discount factor.
Cobb-Doglas utility: analytical solution

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Cobb-Douglas utility: analytical solution

\[ c_t = y_t = h_t = \frac{\beta}{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_{t+1})} \]

- Consumption, output and hours worked, are decreasing in the rate of money growth.
- Intuition: inflation reduces the incentive to work, since the proceeds of work can be used only in the next period.
Cobb-Doglas utility: analytical solution

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Cobb-Doglas utility: analytical solution

\[ p_t = \frac{m_t}{\beta} \left( \beta + \frac{1 - \alpha}{\alpha} (1 + \mu_{t+1}) \right) \]

Price level is proportional to quantity of money.
Cobb-Douglas utility: analytical solution

\[ \pi_t = (1 + \mu_t) \frac{\beta + \frac{1-\alpha}{\alpha} \left(1 + \mu_{t+1}\right)}{\beta + \frac{1-\alpha}{\alpha} \left(1 + \mu_t\right)} - 1 \]

- Inflation is approximately equal to the growth rate of money supply, \( \mu_t \).
- When money grows at constant rate, \( \mu_t = \mu_{t+1} \), then \( \pi_t = \mu_t \).
Cobb-Douglas utility: analytical solution

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\[ r_t = \frac{1}{\beta} \left[ \frac{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_t)}{\beta + \frac{1-\alpha}{\alpha} (1 + \mu_{t+1})} \right] - 1 \]

- Real interest rate can be lowered in this model if \( \mu_{t+1} > \mu_t \), i.e. the rate of growth in money supply in accelerating.
Suppose that money grows at constant rate $\mu$. The seigniorage is:

\[
SE_t = \frac{\mu}{1 + \mu} \cdot \frac{\beta}{\beta + \frac{1-\alpha}{\alpha} (1 + \mu)}
\]
Questions in Monetary Economics: CIA model

1. What causes inflation?
   - Growth in money supply ($\pi \approx \mu$).

2. What determines the amount of seigniorage?
   - Growth rate of money supply ($\mu$) and demand for real balances (preferences).

3. What are the real effects of increasing money supply?
   - No real effects of quantity of money (money is neutral), but faster growth rate of money supply $\Rightarrow$ lower $c_t, y_t, h_t$ (money is not super-neutral).

4. What is the optimal monetary policy?
   - Friedman Rule: $i_t = 0 \Rightarrow \mu_t \approx -\rho$. 
Conclusions

1. Limits to seigniorage.
2. It is not obvious that printing more money can stimulate the economy.
3. Our theory and empirical evidence suggest that faster money growth leads to inflation.
4. Seigniorage could be one reason why central banks do not implement the Friedman Rule.
5. The CIA model is missing some economic channels that can justify discretionary monetary policy (discretionary - central bankers respond to economic conditions).
6. CIA should be viewed as benchmark model, not ultimate answer to all questions related to money and monetary policy.
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