1. Give a proof by contradiction of the following:
   if \( a, b \in \mathbb{Z} \) are such that \( b \) is odd and \( a \) divides \( b \)
   then \( a \) is odd.
   Assume that \( a \) is even so that \( a = 2q \). Since \( a \) divides \( b \):
   \( b = ac = 2qc \). Therefore \( b \) is even, a contradiction.

2. Show that \( n \) is odd if and only if \( 3n \) is odd.
   If \( n \) is odd then \( n = 2p + 1 \). But then \( 3n = 3(2p + 1) = 2(3p + 1) + 1 \). So
   \( 3n \) is odd.
   Conversely (using an indirect proof), if \( n \) is even then \( n = 2q \) so that \( 2 \)
   divides \( 3n \) and \( 3n \) is even.

3. Show by mathematical induction that for all integers \( n \geq 2 \):
   \[
   \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}
   \]
   For \( n = 2 \) one checks that \( \sum_{k=1}^{2} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} > \sqrt{2} \). For \( n + 1 \) one writes:
   \[
   \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}
   \]
   since \( \sqrt{n(n+1) + 1} > n + 1 \) results from \( \sqrt{n(n+1)} > n \)
   which results from \( n(n+1) > n^2 \).

4. Show by mathematical induction that for all integers \( n \geq 2 \):
   \[
   \sum_{k=1}^{n} (k + 1)2^k = n2^{n+1}
   \]
   For \( n = 2 \) one checks \( \sum_{k=1}^{2} (k + 1)2^k = 2 \times 2 + 3 \times 4 = 16 = 2 \times 2^{2+1} \).
   For \( n + 1 \) one writes:
   \[
   \sum_{k=1}^{n+1} (k + 1)2^k = \sum_{k=1}^{n} (k + 1)2^k + (n + 2)2^{n+1} =
   = n2^{n+1} + (n + 2)2^{n+1} = (2n + 2)2^{n+1} = 2(n + 1)2^{n+1} = (n + 1)2^{(n+1)+1}
   \]

5. Consider the sequence defined by \( a_0 = 2, a_1 = 3 \), and
   \( a_{n+1} = 3a_n - 2a_{n-1} \)
   (a) Calculate the next four terms;
   (b) Obtain a formula for \( a_n \) (hint: look at the pattern of \( a_n - 1 \));
   (c) Prove by induction that your formula is correct;
(a) One gets: \( a_2 = 5, \ a_3 = 9, \ a_4 = 17, \ a_5 = 33. \)

(b) Let \( b_n = a_n - 1. \) One has:

\( b_0 = 1, \ b_1 = 2, \ b_2 = 4, \ b_3 = 8, \ b_4 = 16, \) and \( b_5 = 32. \) This suggests \( b_n = 2^n \) and therefore \( a_n = 1 + 2^n. \)

(c) Assume this last formula (which is verified for the first few \( n \)) and consider:

\[
a_{n+1} = 3(1 + 2^n) - 2(1 + 2^{n-1}) = 3 + 3 \times 2^n - 2 - 2 \times 2^{n-1} \\
= (3 - 2) + 3 \times 2^n - 2^n = 1 + 2 \times 2^n = 1 + 2^{n+1}
\]