Chapter One: The Concepts of Game Theory

1.1 Introduction

Game theory has been variously described as the science of strategy or that of conflict resolution. At its core, it has the characteristics of a mathematical construct: a clear set of concepts and assumptions, fundamental theorems, and applications to real world issues. The fact that the issues in question are mostly the domains of the social sciences, however, places game theory in a peculiar position compared to other mathematical and scientific disciplines. Even if one looks beyond the widespread disbelief among social scientists that human and social behavior is amenable to mathematical analysis, the rational choice assumption that lies at the core of game theory leads to a fundamental debate: is the purpose of the theory a description of an independent reality or is it merely a set of prescriptions about what that reality could be in an ideal world of human rationality? Indeed, even the game theoretic concept of rationality is more specialized than conventional wisdom would suggest. Although reason is of course involved in game theoretic rationality, it is focused on the pursuit of happiness by egoistic decision makers. Altruism is hardly a game theoretic motive, but it could very well arise as the outcome of a cost benefit analysis. For the game theorist, every social behavior could be viewed as calculated self-interest.

So, when the President of the United States threatens to veto a piece of legislation introduced by the leadership of an opposing congressional majority, he is not merely engaging in politics, he is also strategizing in some ways. Indeed, the strategy may not just be about the specific legislation in question, it may also be about his reputation and the long term electoral prospects of his party. Undoubtedly, the president received much strategic advice from his aides who collected a flood of information about the issues and their ramifications. Presumably, the whole effort of the presidential team has been to achieve the best possible results given some fairly well understood objectives and priorities. Presumably, the opposing team behaved quite similarly when it formulated the piece of legislation at issue.

When the members of OPEC meet to set their respective quotas of oil production they attempt to limit the world supply in order to raise prices. Evidently, this makes sense only if the decreased output at increased prices yields higher income for OPEC members. But they also meet to deal with another related issue: some of the members are often tempted to take advantage of the increased prices by increasing their production beyond their quota. Of course, prices do decrease somewhat as a result of such unilateral actions, but not as much as if no quotas were set or respected by the majority of the members. Such cheaters are called "free-riders" and they can clearly derive some benefits from their behavior. So, OPEC does not just have to struggle with the negotiation of quotas.
Applicable Game Theory

acceptable by all its members, it also needs to devise plans to deal effectively with the free-rider problem.

The situations just described are the quintessential games that the theory is concerned with. They have fairly well defined decision makers, each with reasonably clear possible courses of action, information, and priorities. Perhaps the greatest uncertainty about that situation is how well each side will do in the end. Often, the risk of miscalculation hangs heavily over every decision maker's head, sometime with dire consequences for themselves and for the whole world.

So, what exactly is the role of game theory? In a purely descriptive approach, it may simply model, as best it can, the complex rules of interaction and information collection of the game, and appraise its actors' preferences. But when it prescribes a "rational" outcome to the problem, the theory requires sweeping assumptions about the degree of the players' rationality. Perhaps these assumptions could be tested by empirical studies. Perhaps they could be graded in terms of the more or less successful, well organized, or highly educated decision-maker. But at some point, the very assumption of rationality inevitably moves game theory from a descriptive to a prescriptive role. Indeed, actual decision-makers can always improve their skills and attempt to reach closer to the ideal of the theory: players who can think in terms of what the others are thinking about what they are thinking... ad infinitum.

Game theory is considered the brainchild of John von Neumann who introduced the first fundamental ideas in 1928. But one can find similar ideas appearing in the work of Augustin Cournot as early as 1838. The first full scale account of game theory is due to von Neumann and Morgenstern in 1944, and a very influential text by Luce and Raiffa appeared in 1957.

Perhaps the greatest individual contribution to game theory is due to John Nash who, in 1950, expanded von Neumann's ideas into the so-called "Nash equilibrium" concept, proved a seminal existence theorem,\(^1\) and partitioned the theory into its two major branches, the "non-cooperative" theory that deals with egoistic self-centered decision makers and the "cooperative" theory that is concerned with the optimal allocation of resources within society. However, the so-called "Nash program" calls for the second branch to be predicated and explained by the first branch. More recently, Reinhart Selten and John Harsanyi introduced important concepts and methods that deal with information and credibility issues that arose from the Nash equilibrium concept. There are other major contributions that I will mention in specific cases.

In these lecture notes, I concentrate on the "modern" approach that privileges the so-called extensive form over the more traditional normal (or strategic) form of the game. This has advantages and disadvantages. The major disadvantage is that existence theorems are somewhat harder to state in an intuitive way. Another one is that it is difficult to speak of strategic equilibrium without some reference to the concept of "beliefs". One advantage is that the scope of the theory is immediately wider, without going through the stages of equilibrium refinements. Another advantage is that it makes effective use of the GamePlan software, which is primarily concerned with the extensive

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\(^1\) In a nutshell, Nash’s theorem says that all reasonably defined games have optimal solutions. His method of proof (using a fixed point theorem) was later adapted to deeper results.
Game theory is now widely applied in economics, business, political science, and international relations. My goal here is to introduce the reader to various such applications with a minimum of technical considerations and requirements. The only prerequisite is a working knowledge of basic college algebra and some understanding of the concept of probability.

1.2 Basic Concepts of Game Theory

Several types of interacting mathematical objects define a game. Only the complete structure, however, makes sense. I will therefore introduce each object type before I define the complete structure as a game. I am only concerned here with the so-called "non-cooperative" theory (a misnomer) that considers independent decision makers in their individual pursuit of happiness rather than the "cooperative" theory that is concerned with the fair or logical distribution of resources within a society of agents.

**Players.** Players are independent decision-makers who influence the development of a game through their decisions. Games have outcomes that the players value according to their personal preferences. The basic tenet of game theory is that players make rational choices that are aimed at achieving the best possible outcomes for themselves. Many game structures require the addition of a special player called Chance (or Nature) who makes some choices according to known probabilities but who has no specific preferences over the outcomes.

**Nodes/States.** As a game proceeds, various states of the world it is describing can be achieved. Such states are also called nodes. Some are final nodes at which the game ends and an outcome is reached. Nodes that are not final call for decisions that will move the game to further, possibly final, nodes. At any non-final node one and only one player has the power to make the next decision that will lead to the next node. That player may be the special Chance (or Nature) player.

**Moves.** At each non-final node, the decisions available to the deciding player are called moves. A move allows the game to evolve from a well-defined state to another and therefore may indicate sequentiality (the exception involves information sets defined below). As far as the rules of the game are concerned, a chosen move cannot be reversed. It only allows one direction of change.

A final node is characterized by the absence of any available further move and, for that reason, does not require a deciding player. Mathematically, it is possible to dispense with the very notion of final node and, instead, to distinguish between final and non-final moves. The latter kind leads to some other decision (or Chance) node while the former kind implies an outcome of the game that can then be specified.

**Outcomes and Payoffs.** Whether it is formally associated to a final node or a final move, an outcome is a development of the game that the players value for itself according to their respective preferences. A standard convention is to represent each player's
preferences by a so-called payoff (or utility) function that associates a numerical score (player specific) to each outcome. The more desired an outcome by a given player, the higher his utility for it.

Although all final moves must involve an outcome that is valued by all players, outcomes may also be associated to non-final moves in the special branch of game theory concerned with so-called "repeated or stochastic games." In that case, outcomes are only temporary and all future (discounted) streams of outcomes are of interest to the players for decision purposes. For that reason, it seems more general and more parsimonious to associate outcomes and their corresponding lists of utilities for all players (their payoff "vectors") to moves rather than final nodes.

Perhaps the simplest meaningful game that can be defined from the above four concepts is illustrated in Figure 1.01 below. Each player controls one state (Node 1 for Player 1, Node 2 for Player 2) and has two possible courses of action at that state (called Up and Down for each). It is clear that the game begins at Node 1 with Player 1 making a choice. But depending on that choice, Player 2 may or may not have a say. If Player 1 chooses Up the game immediately ends in an outcome that is valued at 0 by both sides.

But should Player 1 choose Down, Player 2 then has a choice to make (at Node 2) which will end the game. At each such final move, the upper number is Player 1's utility and the lower one is Player 2's (Up at Node 2, for instance, is valued -1 by Player 1 and 1 by Player 2).

Information Sets. As the game unfolds, players may face decisions with limited information about the exact consequences of their next choice. For instance, two players may make simultaneous decisions, thus being unsure about their opponent's current choice. Since that opponent's current choice affects the development of the game, each side is unsure about the exact consequences of their current decision. This case is
Applicable Game Theory

illustrated in Figure 1.02. The thick dotted line that joins nodes 2 and 3 indicates an
information set for Player 2. It means that when her turn comes, Player 2 is unsure about
whether she is at Node 2 or at Node 3. In particular, it means that she faces exactly the
same choices (Up or Down) at both nodes of the information set.

Another interesting case occurs when a player is unsure about the exact nature of
the game being played, perhaps out of uncertainty about the strengths or preferences of
the opponent. That case is illustrated in Figure 1.03. In that case, the thick dotted line
joining Node 1 and 3 indicates that Player 1 is unsure about what choice Nature made at
the Start node and therefore whether the game is being played in the upper or the lower
part of the game tree.

It has been pointed out that the term "information set" is in fact misleading since their role is to represent a lack of precise information. Perhaps a better term would have been "disinformation" set.
Applicable Game Theory

The Extensive Game Form. A game in extensive form can now be defined by: (1) a set of players; (2) a set of nodes, each belonging to a player or to Chance; (3) a set of moves, each originating from a single node, final if they lead to no other node, non-final if they lead to another (single) node;\(^2\) (4) a set of outcomes associated to final moves, each with a payoff vector made up of a real number for each player; and (5) possible information sets encompassing nodes of a same player, such that any move available to that player at one node has a companion move available at each other node of the same information set. In addition, there must be a single "start" node that is never reached by any move of the game, and the moves must not allow any cycling within the nodes (they must define a single tree).

The last (rather rigid) condition that a single game tree defines a game can in fact be usefully relaxed and complemented by other objects. For instance, there is a branch of game theory that studies "discounted repeated games." In that case, the set of nodes and moves may become infinite, and moves may incorporate a "discount factor" that accounts for the players diminishing concern for the future. A related branch of the theory studies the so-called "discounted stochastic games" that may have only finitely many nodes (or information sets) but allow such states to be re-visited depending on the players decisions. There is a useful representation of such games in "graph" form that I will discuss later. Finally, there is a so-called "normal" or "strategic" form of a game that needs further concepts to define.

1.3 Heuristic Solutions

The fundamental premise of game theory is that players are rational in their decision-making. At the most basic level, this means that they choose each of their moves

\(^2\) Moves originating at a chance node must be further endowed with a probability distribution.
in order to bring about their favorite outcome. But since other players affect the outcome, it is usually not possible for a player to formulate a game plan without some reasoning about what her opponent(s) would rationally do. This reasoning, however, can be rather intricate, depending on the structure of the game.

Let us, however, begin with the simplest kind of structure known as "perfect information" that is illustrated in Figure 1.01: Player 2, if she ever has the turn, would rationally choose Up since this brings her the best score of 1 (instead of -1 if she chooses Down). Expecting this rational choice, Player 1 would contemplate an eventual score of -1, should he choose Down. He would therefore rationally choose Up since that would bring him the better score of 0. Game theory therefore predicts that this game will rationally result in the outcome (0,0).

The thought process of Player 1 that was just described is often called "backward induction" and it succeeds in a class of games, exemplified by that of Figure 1.01. They are characterized by "perfect information:" there are no information sets so that all players always know where they are in the game tree. They can therefore always anticipate all rational future developments and their outcomes for all concerned. In particular, optimal decisions on final moves can be anticipated and, therefore, those on next-to-final moves, and so on, back to the very first available moves. Games of perfect information therefore always have a well-defined rational solution.

Games that lack perfect information are characterized by the existence of information sets. For instance, the game of Figure 1.02 involves three nodes: Node 1 belongs to Player 1 who starts the game while Nodes 2 and 3 both belong to Player 2 and are linked by a thick dotted line that represents an information set. This means that, at her turn to play, Player 2 is unsure of whether Player 1 chose Up or Down at his turn. So, for all practical purposes, the two players make their choices simultaneously. It is essential that Player 2 face the exact same set of choices (Up or Down) at both nodes of her information set. If her choices were distinguishable, she would be able to infer Player 1’s prior choice from what choices are available to her.

We can also investigate here the consequences of the rational choice hypothesis. Player 1 can no longer anticipate Player 2’s decision independently of his own. If he chooses Up, Player 2’s would be better off choosing Up, and if he chooses Down she would be better off choosing Down. But she cannot know what he chose when she makes her own choice. It is essential to understand the meaning of the information set for Player 2: She doesn’t have four possible choices although she has four distinct moves. She only has two choices (Up and Down) since she never knows whether she is at Node 2 or 3 and therefore cannot know which actual move, Up at Node 2 or Up at Node 3, she is making when choosing Up.

The game still has an easy and elegant solution: Player 1 can investigate the consequences for himself of assuming either Up or Down by Player 2. If she chooses Up, he is better off choosing Up since he gets 0 instead of -1. And if she chooses Down, he is again better off choosing Up since he gets 1 instead of 0. So, he is always better off choosing Up no matter what she does! But this rational choice of Up by Player 1 can be

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3 A single node constitutes a “trivial” kind of information set often called a “singleton.” Perhaps heretically I reserve the term “information set” for the case where more than one node is involved.
anticipated by Player 2 who can therefore entertain the belief that she is at Node 2 when her turn comes. She should therefore rationally choose Up at her turn of play. This results in Up and Up and the outcome (0,0). Had she instead entertained the belief that she was at Node 3, her rational choice would have been Down. So, her thinking about Player 1's rationality is essential in her own rational decision.

The game of Figure 1.03 illustrates another important use of information sets. Here, the node called "Start" does not belong to either of the two players. Instead it belongs to Chance (or Nature) who is always a possible neutral player in a game. The moves available at Chance nodes are always given explicit probabilities. Here, the probability of the Chance moves Up and Down are p=0.4 and p=0.6 respectively. It is always assumed that both players know these probabilities. Whether Chance chooses Up or Down, the two players face two very similar game structures (akin to that of Figure 1.01). However, they face different sets of outcomes in the upper and lower parts of the game tree. In addition, Player 1's nodes (1 and 3) are linked by an information set. This means that he cannot observe the actual chance move when he makes his decision. By contrast, Player 2's nodes are not in any information set. So, if and when her turn comes, she will know whether she is at Node 2 or Node 4. This means that she can observe the result of the Chance move in this particular game.

This type of game is called a one-sided (Player 1's side) incomplete information game. Again, we can discuss the implications of the rational choice hypothesis. Since Player 2 will know the situation at her turn of play, she will rationally choose Up at Node 2 and Down at Node 4. This makes the outcomes more predictable for his choice of Up or Down by Player 1. His only uncertainty is now about the chance move. The (final) choice of Down, for instance, should bring him a payoff of -1 with probability 0.4 or a payoff of 0 with probability 0.6. He may therefore expect a -0.4 payoff from that choice. Similarly, his choice of Up will bring him a payoff of 0 (anticipating Player 2's choice of Up at Node 2) with probability 0.4 and a payoff of -1 (anticipating Player 2's choice of Down at Node 4) with probability 0.6. He may then expect a -0.6 payoff from that choice. This makes Player 1 prefer the final move Down to Up, in an expected payoff sense.

Whenever uncertainty is encountered in the form of an information set, a decision at that information set requires an assessment of the chances of being at one or another of its nodes. As we saw in the second example (Figure 1.02), Player 2 could anticipate Player 1's choice of Up (as a dominant choice) and therefore infer that this will place her at Node 2 in her information set when her turn of play comes. In this case, Player 2 uses a forward induction in order to reach an optimal decision.

However, Player 2's inference about Player 1's rational choice of Up was based on an inspection of his payoffs and the realization that he would always be better off with Up, regardless of Player 2's own choice. This will not be true in general, as we will illustrate in the example "Battle of the Sexes" below. In the general case, Player 1's own choice will depend on what he can anticipate from Player 2's choice (in a backward induction) while Player 2's choice still depends on what she can anticipate from Player 1's choice (in a forward induction).

So, games of perfect information that are characterized by the absence of information sets are such that backward induction yields a mutually rational outcome.
However, the backward induction, although grounded in the most solid rational kind of thinking, can yield seemingly "counter-rational" results. Let us now look at an important illustrative example.

The Centipede Game. The game of Figure 1.04 is a (5 nodes) version of the "Centipede" game. Since this is clearly a perfect information game, one may confidently apply backward induction. At Node 5, Player 1 should "take" the outcome pair (4,4). Expecting this, Player 2 would "take" the outcome pair (1,5) since 5 is better than the payoff of 4 she expects from Player 1's choice at Node 5. Similarly, Player 1 will take (2,2) at Node 3, Player 2 will take (1,3) at Node 2, and Player 1 should take (0,0) at the very first turn! Indeed, with those very rational expectations, the game will not last more than one turn and both sides will get a payoff of 0. The paradox is that, should they trust each other a bit, they could move up to Node 3, 4, or 5, where the outcome is always better than 0 for both sides.

![Figure 1.04: The Centipede](image)

Longer Centipede games with payoffs evolving in the same manner can be designed and yield the same conclusion: Player 1 takes (0,0) at his first turn since each players expects the other to "take" at the next turn. Actual experiments with a large enough number of nodes in fact reveals a willingness by real players to take chances in order to reach higher payoffs. Human rationality may therefore not always be well represented by the backward induction.

1.4 Elementary Solution Concepts

We now begin to formalize the solution procedures that were discovered in the last section. We will do so in the limited case of two players and will generalize the ideas to an arbitrary number of players in the next section. We must remember what a choice really is with respect to the concept of information set. We must also remember that games can involve several decision turns. Indeed, games can involve numerous nodes, moves, and information sets. In the Centipede, for instance, Player 1 based his rational decision to "take" at his first turn not only on the expectation that Player 2 would "take" at her first turn but also on the anticipation that both would inevitably "take" at whatever turn they would reach. So, a rational player needs to develop contingency plans in order to address the entire game problem. Indeed, such a player also needs to investigate his
Applicable Game Theory

opponent's own contingency plans in order to exercise rationality to its fullest. This leads to our first concept.

**Pure Strategies.** A pure strategy for a player is a complete contingency plan (a game plan) that specifies in advance what he will do in any development of the game. In particular, a strategy specifies his choices at each of his nodes or information sets.

In very small games, such as the first and second examples, strategies are often confused with moves because each side only faces a single decision turn. But in the Centipede, for instance, each side faces up to three turns of play and must plan what to do if the play develops that far. For instance, Player 1 could plan to pass at Node 1 and to take at each of Nodes 3 and 5 should these ever be reached. This strategy is not necessarily best but it is one possible among many. In fact, Player 1 has eight distinct pure strategies in the Centipede of Figure 1.04 since there are two ways of choosing at each of three nodes 1, 3, and 5, implying 2228 ways for Player 1 of planning for the full game. With only two decision nodes, Player 2 has four distinct pure strategies.

Table 1.01: Normal Form of Figure 1.01

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>Up</td>
</tr>
<tr>
<td></td>
<td>Down</td>
</tr>
<tr>
<td>Up</td>
<td>0.0000E+09</td>
</tr>
<tr>
<td>Down</td>
<td>0.0000E+00</td>
</tr>
</tbody>
</table>

A great advantage of the strategy approach is that it gives rise to a simple and appealing representation of a game called the "normal" or "strategic" form. In the first example, for instance, one may draw Table 1.01 as follows: Player 1's choices Up and Down are listed row by row and Player 2's similar choices are listed column by column. These define four cells in which we list the corresponding outcome pairs deduced from Figure 1.01.

In the row corresponding to Player 1's choice of Up, for instance, the outcome pair is always (0,0) since that is independent of whatever Player 2 was planning at Node 2. In the second row, the outcome (-1,1) results from Player 1's choice of Down and Player 2's choice of Up and (1,-1) is the outcome of Down and Down.

**The Normal Form.** The normal form of a game is a list of pure strategies for each player and the specification of an outcome (in utility terms) for each pair of pure strategies (one strategy per player).

As another example, Table 1.02 below is the normal form of the game of Figure 1.02 obtained similarly. One notable difference is that the outcome pairs in the first row depend on the choice of Player 2 whereas they are independent of it in Table 1.01.

The heuristic solution process of the previous section can now be rephrased in the normal form. In Table 1.01, if Player 1 was to choose Down, then Player 2 could inspect
the last row and conclude that her best reply (to Player 1's choice) is to choose Up which yields her a payoff of 1 (second row in the bottom left cell) instead of -1 (second row in the bottom right cell) if she chose Down. But expecting this choice of Up by Player 2, Player 1 should switch to his best reply Up in order to improve his own payoff from -1 to 0 by moving to the upper left cell. At that point, Player 2 is no longer inclined to move since she can't improve her payoff further by unilateral action. Indeed, should she consider a switch to Down she could expect Player 1 to switch to Down and eventually back to the upper left cell. This discussion suggests the next important concept.

### Table 1.02: Normal Form of Figure 1.02

**The Concept of Best Reply.** A pure strategy for one player is a best reply to a given opponent's pure strategy if it yields his best possible payoff among all his available strategies, provided that his opponent sticks to the given strategy.

In Table 1.01, Up for Player 2 is a best reply to Down by Player 1, Up is a best reply for Player 1 to Up by Player 2, and both Up and Down by Player 2 are best replies to Up by Player 1. So, a best reply to a given strategy is not necessarily unique, but when strategies in a pair such as Up and Up in Table 1.01 are both best replies to each other, a very rational way of playing the game seems to have been reached. This is our next concept, one that is very elegant in its simplicity.

**An Elementary Concept of Equilibrium.** Equilibrium is a (pure) strategy pair such that each strategy is a best reply to the other.

In the normal form of Table 1.02, for instance, the strategy pair (Up,Up) is an equilibrium since Players 1 and 2 find it best to respond to Up by Up.

In each of the four games we have visited so far, we have been able to identify a single rational pair of strategies that clearly meets the conditions of equilibrium. This is unfortunately not the general case. As we will see, there doesn't always exist equilibrium in pure strategies. However, we will extend the pure strategy concept to the two (almost equivalent) concepts of mixed and behavioral strategies. In that extension, equilibrium will always be guaranteed to exist. But uniqueness is never going to be a standard feature of game theory.

Although the normal form has great appeal it may be cumbersome in some games. To illustrate this fact, consider again the Centipede of Figure 1.04. If one denotes by T and P the choices of Take and Pass, and by PTT the strategy described above (pass, then
take and take), for instance, the normal form of this Centipede contains 84 entries and is shown in Table 1.03.

Table 1.03: The Normal Form of the Centipede

The entire upper half of this table gives details that appear entirely irrelevant to a rational analysis of this game. All strategy pairs in this upper half correspond to Player 1 immediately taking the outcome (0,0) seemingly making all subsequent plans largely irrelevant. In fact, there is worse: one can find eight distinct equilibriums (in pure strategies) in Table 1.03. For instance, the pair (TTP, TT) is an equilibrium since neither side can improve upon it by unilateral action (TTP and TT are best reply to each other). But there is something odd in this statement: if Player 1 is really planning to pass at Node 5, Player 2 should not be planning to take at Node 4 according to the logic of backward induction! The reason that this happens is that the planning by Players 1 and 2 to take at their first turns makes all subsequent developments truly irrelevant in this analysis. So, the logic of backward induction does not seem to square well with that of the normal form equilibrium analysis. This is one reason why the extensive form analysis is usually finer than the normal form one. Yet, the normal form has been more publicized than the extensive form, especially in the non-specialized literature.

1.5 A General Solution Concept

Let us now elaborate on some statements made earlier: is there always a single pure strategy equilibrium in a game? To understand the issues, let us consider the well-known game of Heads and Tails. There are two players, each holding a coin, which they

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4 This is what motivated the development of so-called “perfect” equilibrium concepts that we will study later in this chapter.
Applicable Game Theory

may show with heads or tails facing up. They simultaneously make a choice of heads or
tails and show their coins to each other. If both coins show the same side, Player 1 wins
(i.e., Player 1's payoff is 1 and Player 2's payoff is -1), if they show different sides, Player
2 wins. The normal form of this game is given in Table 1.04 below.

<table>
<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>1.0000E+00</td>
<td>1.0000E+00</td>
</tr>
<tr>
<td>Tail</td>
<td>-1.0000E+00</td>
<td>-1.0000E+00</td>
</tr>
</tbody>
</table>

Table 1.04: Heads and Tails

This is a case of so-called zero-sum game where whatever one player wins is what
the other loses. Continuing our earlier best reply analysis, one sees that Head for Player 1
is best against Head for Player 2, which is best against Tail for Player 1, which is best
against Tail for Player 2, which is best against Head for Player 1 ... back to the
beginning! Indeed, there is no equilibrium in pure strategies in this game (this has
nothing to do with the zero sum condition). But if childhood experience means anything,
one should be as unpredictable as possible in this game and perhaps simply toss the coin
rather than make any specific choice. Thus heads and tails would appear with equal
probability p=0.5. So, suppose that Player 2 adopts this probabilistic behavior and let us
investigate its consequences for Player 1. When contemplating the choice of Head and if
aware of Player 2's plan, Player 1 can make the following expected payoff calculation:
half the time he will meet Head by Player 2 and the other half he will meet Tail. So half
the time he will get a payoff of 1 and the other half the time he will get -1. So, he will
have an expected payoff of 0. By a similar calculation this is also his expected payoff
when he contemplates the choice of Tail. So, both Heads and Tails are best for Player 1
against the toss of the coin by Player 2. Indeed, Player 1 can do just anything he wants, if
Player 2 sticks to this toss of the coin plan, Player 1 will get 0 in expectation. So, Player 1
may as well adopt the same plan since it is one among the best replies. By a symmetric
calculation, Player 2 can do anything in response, including tossing the coin. So, we have
reached a new kind of equilibrium (best reply to each other) that relies on probabilistic
rather than deterministic strategies. We now formalize this.

Mixed Strategy: A mixed strategy for a player is a randomization of his pure
strategies. In other words, it is given by a probability distribution over his available pure
strategies. Of course, a pure strategy can be viewed as a particular case of mixed strategy
when its probability is one and all others are zero.

Before we formalize the expected payoff concept, we need to expand our
definitions to the case where more than two players are involved. Let us consider an
example first. The game of Figure 1.05 is a three-player game in extensive form.
Figure 1.05: A Three-Player Game

The normal form of Figure 1.05 is a bit more complicated than the tables we have seen so far. It requires three dimensions (thereby the players' names Row, Column, and Depth). Table 1.05a shows the outcomes when Depth chooses Front and Table 1.05b shows them when Depth chooses Back.
The solving of this game is a little more complicated than the ones we have seen so far. It will turn out that Row prefers Down to Up since he gets a payoff of 1 with Up but will get a strictly better expected payoff from Column's and Depth's rational behavior. Solving with methods that we will study later points to Column using a mixed strategy defined by Left with probability 1/3 and Right with probability 2/3, and Depth using a mixed strategy defined by Front with probability 2/3 and Back with probability 1/3. Row's expected payoff from these mixed strategies is simply the sum of the payoffs he would receive from each of the four possible combinations of Column's and Depth's pure strategies weighted by its corresponding probability. For instance, (Left, Back) has probability $(1/3) \times (1/3) = 1/9$, it yields a payoff to Row choosing Down of 3, and contributes $(1/9) \times 3 = 1/3$ to Row's expected payoff of choosing Down which, when computed fully, adds up to 5/3. This idea can be made more formal.

**Expected Payoff.** The expected payoff for a player resulting from mixed strategies (one for each player of the n players of the game) is the sum of the payoffs associated to each of the possible pure strategy combination times the probability of that combination as defined by the mixed strategies.

Since mixed strategies involve probabilities of using pure strategies, it is intuitively clear that they implicitly define probabilities of using the moves involved in those pure strategies. Figure 1.06 shows how the probabilities involved in the game of Tables 1.05a, and b, translate into move probabilities in Figure 1.05. The probability of the move Back at Node 3 or 4 is denoted $p=0.33333$, for instance. Of great interest for future developments are the probabilities denoted $b=0.33333$ and $b=0.66667$ of Depth being at nodes 3 and 4 resulting from these strategies. Such probabilities will later be called beliefs.

**Behavioral Strategy.** A behavioral strategy for a player is given by probability distributions over his available moves at his available nodes or information sets.

In games more complex than that of Figure 1.05, each pure strategy would likely correspond to several moves at several nodes and information sets, and the probabilistic
Applicable Game Theory

relationship between moves and strategies could be less transparent. In fact, it could even be ambiguous when the information structure does not conform to a principle known as perfect recall.\(^5\) When that principle is satisfied mixed and behavioral strategies become equivalent. The former are the natural vehicles of rationality in normal form games and the later play the same role in extensive form games with the possible complement of beliefs. Most of early game theory simply assumed perfect recall and only dealt with the normal form and the mixed strategy framework. This led to a fundamental theorem obtained by John Nash in 1950.

The Nash Equilibrium in Normal Form Games. It is now clear that the premise of rationality, in the case where several independent decision makers are involved, leads to the concept of equilibrium, a situation in which the player are making their best possible strategic choice given their opponents’ own expected choices. It also appears that the state of equilibrium may require that players contemplate probabilistic choices and therefore use mixed strategies. And the mixed strategies yield expected payoffs that can be obtained from the normal form of the game by simple algebra. In this perspective, the central question that needs to be addressed is whether games always have an equilibrium. Formally:

**Definition 1:** Let \(s_1, \ldots, s_n\) be mixed strategies, one for each of the \(n\) players (\(n \geq 2\), finite) of a game in normal form. Let \(P(s_1, \ldots, s_n)\) be Player \(i\)'s expected payoff from these strategies. Then:

1. \(s_i\) is a "best reply" for Player \(i\) to the \(n-1\) strategies \(s_{-i} = \{s_j, j=1,\ldots,n, j \neq i\}\) if there is no strategy \(\sigma_i\) (for Player \(i\)) such that \(P(\sigma_i, s_{-i})\) is strictly greater than \(P(s_i, s_{-i})\).
2. if \(s_i\) is a best reply to \(s_{-i}\) for each player \(i=1,\ldots,n\) then the strategy profile \(\{s_1,\ldots,s_n\}\) forms a Nash Equilibrium of the game.

Now, the fundamental question of existence of game equilibrium raised above finds a simple and powerful answer:

**Theorem 1:** Any game in normal form admits at least one Nash equilibrium.

For this finding, John Nash was awarded the Nobel Prize in Economics. His mathematical proof involved elegant but advanced mathematics.

1.6 Advanced Solution Concepts

Nash's result is closely tied to the normal form. But as we observed with the Centipede game, the normal form may include irrelevant information (detailed outcomes of equivalent strategy pairs), may leave out some relevant information (such as timing), and may produce Nash equilibria that are at odds with the logic of backward induction. In fact, one need not limit the argument to backward induction, which is only relevant to games of perfect information. Indeed, looking back at Figure 1.05, which has imperfect information, one would find the Nash equilibrium illustrated in Figure 1.07 below.

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\(^5\) Loosely speaking, a game has perfect recall if information sets do not entail any ambiguity about prior information and developments.
The logic of this equilibrium is characteristic of one failure of the normal form. Since Row ends the game with Up on the first turn, Column and Depth never have a chance to use their strategies. So, a switch by Column from Left to Right, or a switch by Depth from Front to back, will have no effect on their payoffs. The choices of Left and Front shown in Figure 1.07 are best by default. But should one pursue the analysis a step further, a gross inconsistency appears: expecting Front, Column should go Right. The logic of game theory seems to stop at Node 2 simply because it is never reached. This prompts the definition of a critically important concept.

The equilibrium path is the set of nodes and information sets that can be reached with positive probability from the start node when play is according to the given equilibrium.

In Figure 1.07, for instance, the equilibrium path is reduced to Node 1 (the start node, which is always reached). So, the inconsistency detected in Figure 1.07 will likely be encountered when a node or an information set is off the equilibrium path. At such a node, what is prescribed by the equilibrium can possibly be an illogical plan provided it doesn't make the node sufficiently attractive to change the prior decision to not reach the node. If it did, then the node would be admitted into the equilibrium path and the inconsistency at that node could no longer be sustained. To remedy such potential inconsistencies, Selten offered the following concepts.

A subgame of an extensive form game is any part of the game that forms a complete game in its own right. In particular, it must have a start node and any of its information sets must be entirely contained in the subgame. In Figure 1.07, for instance, nodes 2, 3, and 4, and the information set between nodes 3 and 4, the players Column and Depth (and indeed Row as well), the moves Left, Right, Front, and Back, and the final payoffs to Column and Depth, after Depth's move, form a subgame.

A subgame perfect equilibrium of a game is a Nash equilibrium that defines a Nash equilibrium in all of its subgames.
Put another way, the decisions prescribed by a subgame perfect equilibrium should be optimal for each player on as well as off the equilibrium path. Unfortunately, this is not quite so since not all isolated nodes (singletons) are the start node of a subgame (see homework problem #8). Evidently, the subgame perfect equilibrium reduces to backward induction in perfect information games, but it clearly generalizes it to games such as that of Figure 1.05 and rules out the Nash equilibrium of Figure 1.07. It is not too difficult to generalize Nash's methodology to establish that any game (as they have been defined so far) admits a subgame perfect equilibrium.

Although Selten's ideas have been rather successful, they did not eliminate all the inconsistencies of the normal form. At about the same time (mid 1960's) as Selten was making his contribution, Harsanyi was working on an entirely different type of issue. In many actual games, players may know the rules as well as their own preferences on the various outcomes, but may not be quite sure about the other players' own preferences. Harsanyi's conceptual contribution was to turn this "incomplete" information problem into one of "imperfect" information by an astute use of chance moves and information sets. The game of Figure 1.03, for instance, is a typical incomplete information game where Chance (Nature) makes an initial random choice with specified probabilities between two possible games with the same decision structure but different players' payoffs. Of course, there could be many more payoff configurations and corresponding initial chance moves. In fact, one can imagine that each player has several possible "types" (meaning preferences). And there are given probabilities that each type of one player will meet each type of the other(s). In Figure 1.03, for instance, Player 2 has two types and only she knows which type she is at her turn of play. Player 1, by contrast, has only one type here but is unsure about whom he is facing.

Games such as that of Figure 1.03 can easily be turned into normal form games by listing all strategies for each player (in particular what she will do depending on her type), and the expected payoffs for all strategy profiles taking all type probabilities into account (see homework problem #1). In that approach, the type probabilities seem to be set in stone. In particular, there is no apparent learning of the players about each others type from the observation of each others' moves. To understand this issue, consider the game of Figure 1.08. Player 2 doesn't know Player 1's type, which is decided by a chance move at the outset with the given probabilities. These are called initial --or prior-- beliefs. But once the two sides choose their strategies, there are corresponding move probabilities at each of the nodes (for Player 1) and at the information set (for Player 2). One solution of this game is pictured in Figure 1.09.
Applicable Game Theory

Figure 1.08: Learning the Opponent's Type

Player 1's behavior is of course dependent upon his type. In Figure 1.09, he goes Up with certainty in the upper part of the tree and does so only with probability $2/3$ in the lower part. Player 2 meanwhile is planning to use her two possible moves each with 50% probability. What kind of rationality is underlying such plans? The critical concept here is that of beliefs, in particular those appearing on top of Player 2's nodes.

Beliefs are probability distributions over the nodes of each information set. Beliefs are consistent with strategies if they satisfy Bayes' Law (from the theory of conditional probabilities), assuming that move probabilities are independent events.
In Figure 1.09, beliefs are consistent. The probability \( p \) of reaching Node 2 is the probability of going from Node 1 to Node 2 times that of being at Node 1 in the first place. This means \( p = 1.0 \times 0.4 = 0.4 \). Similarly, the probability \( q \) of reaching Node 4 is that of going from Node 3 to Node 4 times that of starting at Node 3. This means \( q = (2/3) \times 0.6 = 0.4 \). Since \( p = q = 0.4 \), there are equal chances of being at Nodes 2 and 4 within Player 2's information set, thus resulting in beliefs of 0.5 for each node. These last beliefs are called "updated" --or posterior-- beliefs about the type of opponent Player 2 is really facing. A standard interpretation is that Player 2, when observing the move Up by Player 1, is learning about his type.

Of course, the updated beliefs are critical in justifying Player 2's plan of using Up or Down with 50% chances. Indeed, it is only with the given beliefs that those two choices are equally attractive for Player 2. In turn, Player 2's plan determines expected payoffs for Player 1 (\( E = 0.0 \) under both Nodes 2 and 4). These expected beliefs then make it possible for Player 1 to make the specified choices optimally. This equilibrium is therefore a subtle balance between move probabilities and beliefs so that both sides are indeed doing their best. This equilibrium concept is called perfect Bayesian.

A perfect Bayesian equilibrium is made up of a strategy for each player and a set of beliefs over all information sets such that the beliefs are consistent (with strategies according to Bayes' Law wherever applicable) and the strategies are optimal given the beliefs.

A perfect Bayesian equilibrium is clearly also a Nash equilibrium since the optimality of the strategies will translate back into the normal form of the game. Indeed, a perfect Bayesian equilibrium is also subgame perfect but it goes a little deeper into the rationality analysis in games such as that of Figure 1.08. Indeed, that game has no (proper) subgame. However, there are still some inconsistencies that need discussion. Consider another solution of Figure 1.08 pictured in Figure 1.10.

This is technically a perfect Bayesian equilibrium. However, there is something odd about the 50% beliefs at Nodes 2 and 4. They do not result from the same application of Bayes' law as in Figure 1.09 for the simple reason that there is zero probability of ever getting to either nodes, given Player 1's choices at Node 1 and 3. Indeed, Node 2 and 4 are off the equilibrium path and Bayes law cannot determine beliefs about them. So they can as well be arbitrary, as they are here. One can imagine the potential contradictions of optimality calculations based on completely arbitrary beliefs off the equilibrium path. This is not unlike the difficulty that led to the subgame perfect concept.
Kreps and Wilson proposed a remedy in the early 80's. It relies on a simple way to ensure that Bayes' law can always be applied. Imagine that there is an extremely small but distinct probability that a mistake will happen in the implementation of strategy. Then Player 1 could possibly find himself going Up rather than Down at each of Nodes 1 and 3 (and not necessarily with the same probability). For instance, assume that there is a probability $p=0.00006$ of Up (and therefore $p=0.99994$ of Down) at Node 1, and a probability $q=0.00004$ of Up (and therefore $q=0.99996$ of Down) at Node 3. Then Bayes' law applies and gives the correct 50-50 beliefs at Nodes 2 and 4. Now let the probabilities $p$ and $q$ approach zero in the same proportion as above (for instance, $p=0.000000006$, and $q=0.000000004$ is one such step along the way). Each time the beliefs at Nodes 2 and 4 remain valid. As the probabilities $p$ and $q$ vanish into the infinitesimal, the beliefs off the equilibrium path remain determined by Bayes' law and are therefore consistent. So, the equilibrium of Figure 1.10 is not so absurd after all. Such a limit process on the strategy-belief pairing is often called "trembling" and the resulting equilibrium is called sequential.

A sequential equilibrium is the limit of perfect Bayesian equilibria as trembling approaches zero.

As always, a concept is only useful if it has the merit of guaranteeing existence, which was proved by Kreps and Wilson. However, the above example might give the wrong impression that a perfect Bayesian equilibrium is always sequential since we were able to justify the solution of Figure 1.10 by a trembling type of argument. The example of Figure 1.11 should dissipate this impression.
Applicable Game Theory

Figure 1.11: A Non-Sequential Perfect Bayesian Equilibrium

This is just the game of Figure 1.08 with Player 1's nodes joined in an information set. But the trembling argument that rescued the equilibrium of Figure 1.10 can no longer apply. The reason is that Up is now a single choice for Player 1 at both nodes 1 and 3. So, whatever trembling applies to the move Up at one node applies to the move Up at the other. But if so, the probabilities of reaching nodes 2 and 4 will always be in the proportion 0.4/0.6=2/3 in any trembling. This can never be reconciled with the 50% beliefs at node 2 and 4 and the equilibrium is therefore not sequential although it remains perfect Bayesian.

1.7 Advanced Game Structures

Until this point, extensive game forms have been defined on so-called game trees characterized by the existence of one start node, by the condition that any other node has exactly one corresponding move to enter it, and by the prohibition of any cycle that would allow to return to a node following consecutive moves. In addition, such game trees have implicitly had only finitely many nodes. A game defined on such a finite game tree has therefore a predetermined maximum number of turns. In many instances, this is a prohibitive limitation.

Consider, for instance, the following game that could be played by three friends sitting around a table. One envelope contains a $1 and a $2 bill. Each person can either take the envelope or pass it to their left. Whoever takes the envelope ends the game, keeping the $1 bill and giving the $2 bill to the friend on their right. So, each player’s favorite outcome (which yields $2) is to pass to his left and see that person take the envelope. The next best (which yields $1) is to take the envelope at his turn. And the worst (yielding $0) is to see the person on his right take the envelope instead of passing it. The initial player can be chosen at random. Figure 1.12 is a possible representation of that situation and it is evidently not a game tree.
What rational plan could possibly arise from such a situation? Perhaps one player (say Blue) could decide to take at some point. But if Green anticipates that, she will pass. But anticipating that, Red should take. Then, Blue's plan to take would be irrational. By symmetry, it seems irrational for anyone to take at any point, at least with certainty. A probabilistic plan actually doesn't do any better.

However, something critical is missing from the representation of Figure 1.12. The three friends would evidently not like to sit around that table until they grow old for such a mediocre prize. Either they would feel some cost associated with sitting at the table for too long a time, or the prospect of getting any prize would feel less and less attractive as the time it takes to get it increases. In fact, both of these approaches that account for the element of time in reaching an outcome can be successfully incorporated into Figure 1.12. In the first case, a cost (some negative quantity) is associated to the "pass" moves and eventually accumulates in a prohibitive way. In the second case, a discount factor is added to the pass moves in order to represent the declining interest of the players for future outcomes that take too long to achieve. Technically, this example therefore suggests three possible generalizations of the game tree approach:

(i) Games can be defined on graphs rather than trees, as in Figure 1.12;
(ii) Non-final moves can involve payoffs to some or all of the players; and
(iii) The future can be discounted.

Figure 1.13 below shows the game of Figure 1.12 with an added cost (-0.1) for each of the 'pass' moves incurred by the corresponding player. The unique solution is for the players to "take" with probability \( p=0.316228 \) (actually \( \sqrt[0.1]{} \)).
Intuitively, each player sees the prize ($2 or $1) melt away as the game goes on. It therefore makes more sense to "take" early enough in order to obtain a positive prize.

Conceptually, discounting has two possible interpretations. Figure 1.14 illustrates one interpretation: the blue player is considering to move from Step #N to Step #N+1. If the move is taken, there is a chance that the game will stop and result in an outcome of (-5) for each player. The probability that this happens is p=0.2 and the probability that Step #N+1 will be reached is therefore p=0.8. From an expectation viewpoint, this is the same as incurring a payoff of 0.2×(-5)=-1 when taking the move and pre-multiplying by the "discount factor" 0.8 whatever payoff is expected from reaching Step #N+1. This "discounting" is shown in the lower part of Figure 1. Now "move" involves a payoff (-1) to each side (although it is not a final move) and the discount factor d=0.8 (above the move name) is applied to any expected payoff at Step #N+1.
Another interpretation of discounting is that players value the future less than they value the present. The farther the future the lower is the value. So, tomorrow can be discounted by a factor $d$ positive and less than one. This means that whatever payoff could be realized tomorrow is pre-multiplied by this factor $d$ when viewed today. Similarly, the day after tomorrow is twice discounted so that its anticipated payoff is pre-multiplied by the square of $d$ when viewed today. And any payoff anticipated in $n$ turns from now is pre-multiplied by $d^n$ (the $n^{th}$ power of $d$) when viewed today. We are now ready for a more general definition of games.

A Stochastic Game is defined by a graph instead of a tree. It can therefore allow cycling (i.e., a set of consecutive moves such that the end node coincides with the start node). Non-final moves can involve both payoffs and discounting.

I will place the following conditions on the stochastic games considered in these notes: (1) There are finitely many players, nodes, and moves; (2) any cycle must involve at least one move that is discounted; and (3) there must be at least one singleton from which any node of the graph can be reached.

In practice, as in Figure 1.12, a (single) start node will stand out by lacking any move that leads to it. The discounting assumption then allows the very definition of a corresponding normal form. A pure strategy for a player is simply the specification of what the player will do (with certainty) at any of his singletons and information sets. A (pure) strategy profile is the choice of a (pure) strategy for each player.

To see why such a profile defines a true payoff for each player, one may reason as follows: From the start node, a move (or probabilistic moves if the node belongs to Chance) is defined and may involve payoffs to some or all of the players. If the move leads to another node, each player can expect a total payoff made up of today's payoff plus a possibly discounted expected payoff from the node the move leads to (or a
probability distribution of such when Chance is involved). Of course, the expected payoff from the node the move leads to is similarly obtained. When one iterates the process, only two things can happen: (1) it ends at a final move with payoffs to everyone that are added to any and all other incurred along the way (with possible discounting); (2) Some cycling is involved. In this last case the following ambiguity seems to develop: the first node of the cycle that is encountered involves an expected payoff whose calculation eventually returns to that very node! This is where the discounting assumption becomes critical. Because at least one move is discounted, the expected payoff at that node (call it e) involves possibly some payoffs along the way (call them u) plus that same expected payoff discounted by some factor (or product of factors) d<1. Thus e=u+de or e=u+(1-d).

So, expected payoffs are well defined. In fact they are well defined for any behavioral strategy profile by a similar argument.

Expected Payoffs (in a stochastic game) for a player, resulting from a behavioral strategy profile, are defined at each node by the condition that the expected payoff at any given node is the sum of the payoffs from each move available at the node plus the discounted expected payoff of the node each move leads to, weighed by the probabilities of taking each of the moves.

Once expected payoffs are unambiguously associated to any strategy profile the normal form is well defined and the concept of Nash equilibrium becomes applicable. But this concept is clearly dependent upon the identification of a start node. Therefore, it is vulnerable to the same criticism that led to the perfection concepts in game trees: if a node or information set is off the equilibrium path, what an equilibrium strategy specifies may not be optimal there. Subgames could be defined just as in game trees. But in a graph, there may be even fewer subgames than in game trees simply because of the existence of cycles. A more promising candidate is the perfect Bayesian equilibrium or even the sequential one. A standard approach in the literature is to define a so-called Markov perfect equilibrium. But there is no consensus about what exact restrictions must apply to distinguish it from the Nash concept. Again, claiming no generality, I choose here to generalize the perfect Bayesian equilibrium concept.⁶

A Markov perfect equilibrium in a stochastic game is a behavioral strategy profile and a set of beliefs such that probable moves are optimal at singletons and information sets given the expected payoffs and the beliefs, and beliefs are consistent with strategies according to Bayes' law wherever applicable. A sequential equilibrium is a limit of Markov perfect equilibria as trembling approaches zero.

The existence of Markov perfect and sequential equilibria in stochastic games is proven in appendix under the conditions given above. It should be noted that, when using such a concept, one could dispense with the requirement of a start node entirely. Indeed, any singleton can be viewed as a possible start node of the game. It would be necessary to specify which of them is the start node when using the Nash concept, but it is superfluous with the Markov perfect concept. In Figure 1.12, for instance, if one removes the start node S and its associated chance moves the resulting structure still qualifies as a stochastic game and has three singletons that can each serve as potential start node. It is

⁶ Most authors actually choose to generalize the subgame perfect equilibrium concept.
Applicable Game Theory

no longer clear where the game starts but play is optimal at any singleton in any Markov perfect equilibrium.

Stochastic games can also be used to model repeated games although they limit the solutions to the so-called Markov strategies. Consider, for instance, the Battle of the sexes game of Chapter 1. One could imagine playing it just once. But most couples face this kind of decisions over and over again. Indeed, if they are really committed to each other, the game has an essentially open-ended future. And both are likely to value the present as well as their future in some proportion. A standard approach to modeling this kind of situation is to define a discounted repeated game where the given "one-shot" game is repeated, possibly forever, but where the future developments are discounted by a given factor d. Figure 1.15 below is a typical stochastic game representation of this situation.

Figure 1.15: The Repeated Battle of the Sexes

After each turn of the game, depending on the way it has been played, the players receive their respective payoffs and move on to the next turn. Since there are four different ways the game can be played at each turn, I distinguish four different "start nodes" for each turn. In the upper-left corner, for instance, it is assumed that both sides chose the Ballet at their previous turn. Indeed, all (red) moves leading to the blue node Ballet-Ballet involve the payoffs (1,0) of the one-shot Battle of the sexes. But they also involve the discount factor d=0.9. This is because the players value the next turn with this discount factor and appraise their planned move according to what payoff they immediately get and what expected (and discounted) payoff they will get when reaching the next turn. Of course, the blue moves are not discounted because the players move simultaneously and therefore discount only once both sides have made their choice and received their corresponding one-turn payoffs.
One can add a single "starting" one-shot game to Figure 1.15 with a well-defined start node that could never be returned to. But this does not change anything to the discussion of the solution. Because only four states of the game are distinguished in Figure 1.15 (Ballet-Ballet, Ballet-Fight, Fight-Ballet, and Fight-Fight), and because this is a stochastic game, we can only obtain so-called Markov strategy solutions. But they are often instructive of the differences between playing a game once or repeatedly. For instance, the one-shot Battle of the sexes game had only two pure strategy equilibria where one side was taking advantage of the other. Those "selfish" equilibria reappear in the game of Figure 1.15. Playing Ballet-Ballet all the time is indeed a Markov perfect equilibrium of this game. But there are other solutions with more fairness. In Figure 1.16, for instance, the players carefully alternate between going together to one event or the other.

Figure 1.16: A Fair Solution

Expected payoffs vary according to what has just been done at each turn. She is obviously better off when they both went to the Fight since she is expecting a joint trip to the Ballet at the next event. This is reflected in the expected payoffs (E=5.26, E=4.74) that appear under the blue node Fight-Fight. But he is also doing quite well because of his expectation that after the next event Ballet-Ballet, he will be expecting E=5.26 that appears (in red for him) under the blue node Ballet-Ballet. Note that the upper-right and bottom-left parts of the game are off the equilibrium path. But play according to the Markov perfect equilibrium shown here is still optimal there. At the Ballet-Fight blue node, for instance, she is planning Ballet with an expected payoff E=4.74 (under node C21) that is clearly better than E=2.74 (under node C22) she would get by going to the fight. Likewise, expecting this, he believes to be at node C21 when his turn comes and will do better going to node Ballet-Ballet. Indeed, he expects 0 from the move plus
Applicable Game Theory

\[ dE = 0.9 \times 5.28 = 4.74 \] from choosing Ballet and only -1 plus \[ dE = 0.9 \times 4.73 \], for a total of 3.26, from choosing Fight.

Stochastic games and the discounted repeated games they can model are a great modeling resource that is dramatically under used in the literature. One reason is that equilibria are difficult to obtain analytically and systematically. Another reason is that the number of valid solutions with completely different interpretation tends to explode as one allows more details in the possible states the game can be in.

Homework

1. Write the normal form of the incomplete information game of Figure 1.03. Hint: Player 1 has two strategies but Player 2 has four.

2. Find all eight pure strategy equilibria of Table 1.03 and discuss any discrepancy they exhibit with respect to backward induction in Figure 1.04.

3. Consider the normal form of Table 1.06 below. Draw two distinct extensive form games, one with perfect information, and one with one information set, that both have this normal form. This example shows that a given normal form can correspond to several extensive forms. Conversely, it is not difficult to imagine that something essential can be lost in the writing of the normal form of a game.

\[
\begin{array}{c|c|c}
\text{Player 1} & \text{Player 2} & \text{Payoffs} \\
\hline
\text{Up} & \text{Up} & 0.00 \quad 0.00 \\
\text{Up} & \text{Down} & 0.00 \quad 0.00 \\
\text{Down} & \text{Up} & -1.00 \quad 1.00 \\
\text{Down} & \text{Down} & 1.00 \quad -1.00 \\
\end{array}
\]

Table 1.06

4. Can you modify the payoffs in Figure 1.05 (the Centipede) so that both sides always "pass" but the final payoff is worse than taking at the first turn? Explain your answer carefully.

5. The Battle of the Sexes. The story, as told by Luce and Raiffa in their famous text Games and Decisions goes as follows (p. 91): "A man (He), and a woman (She) each have two choices for an evening's entertainment. Each can either go to a prizefight or to a ballet. Following the usual cultural stereotype, the man much prefers the fight and the woman the ballet; however, to both it is more important that they go out together..."

Assume that both sides' worst outcome, is when they go their separate ways each to the other's favorite event, and is valued at -2. The next worst is when they go their separate ways, but each to their own favorite event, valued at -1. The best outcome, for each, is when their partner joins them in their favorite event (valued at +1) and their next
best is when they join their partner in his or her favorite event (valued at 0). Construct a normal form table, and draw an extensive form version of the Battle of the Sexes. Find all pure equilibria.

6. Draw the extensive form of the following game: Chance first flips a fair coin. If Heads, Player 1 decides whether to "take," thus ending the game with zero payoffs for everyone, or to "pass." If Tails Player 2 decides whether to "take," thus ending the game with payoffs (-1,0,1) in that order, or to "pass." If either Player 1 or Player 2 passes, Player 3 has the move but does not know who gave it to her. She may either go Up or Down. If she goes Up after Player 1, the outcome to the three players is (-1,0,1). If she goes Down after Player 1 the outcome is (0,-1,1). If she goes Up after Player 2, the outcome is (0,-1,1). And if she goes down after Player 2, the outcome is (1,0,-1). What solution can you predict?

7. Selten's Horse. The following game is known as Selten's horse from the name of its author.

Figure 1.17: Selten's Horse

Write the normal form version of this game and find two pure Nash equilibria.

8. Consider the game and its solution pictured in Figure 1.18 below. Verify that it is a Nash equilibrium and describe the equilibrium path. Explain why this Nash equilibrium is also subgame perfect although not all play off the equilibrium path is optimal.
9. Modify the game of Figure 1.12 by applying a discount factor $d$ to each of the three "pass" moves. Solve and interpret.

10. Consider the game of Figure 1.19. Describe a situation or a simulation that could be modeled by such a game. Solve for Markov perfect equilibria and interpret the solutions.

Figure 1.19: A Not So Perfect Equilibrium