Incomplete Information

Two versions of the most basic deterrence game between two players called Challenger and Defender are depicted in Figure 5.01. The challenger (in blue) can either "challenge" the status quo or "stay" in the current situation. In turn, the defender (in red) can submit to the challenge or escalate the dispute. In real life instances, this structure arises when a start-up business decides to enter a market dominated by an established incumbent firm, or when a nation state challenges a status-quo power over the control of a valuable asset.

In all cases, the challenger prefers that the defender submits since it would yield a successful market entry or the acquisition of a desirable prize. But in the upper part of the figure the game is between a defender and a strong challenger, one who would win a fight and prefer it to a simple surrender. In the lower part of the figure, however, the defender is facing a weak challenger who expects to lose and therefore fears a fight. The solution of each game is easily obtained by backward induction: a strong challenger challenges and sees the defender submit while a weak challenger stays expecting the defender to fight.

Uncertainty changes the game profoundly. In Figure 5.02, the defender is uncertain about the challenger's type. With equal probability he may be facing one or the other.
The only solution of that game, a perfect Bayesian equilibrium (PBE), is shown in Figure 5.03.
In this case, the weak challenger challenges with some probability ($p = 0.5$) and the defender fights with some probability ($p = 0.5$). The weak challenger is therefore taking advantage of the defender's uncertainty to possibly extract concessions while taking the risk of a fight. Note the updated beliefs of the defender when reaching his turn. The initial beliefs are 50-50 but they change to a 2/3 updated beliefs of facing a strong challenger.

A game structure such as Figure 5.02 is often called a "signaling" game because it gives the first player an opportunity to signal his type through his initial choice. Of course, such a player can engage in some form of bluffing by taking advantage of the other side's uncertainty. Indeed, if the defender's initial belief that the challenger is strong is high enough, there may be no risk at all and a weak challenger can take full advantage of the situation:

![Figure 5.04](image)

In Figure 5.04, the initial belief (of the defender) that the challenger is strong goes up to 0.9. As a result, the only PBE sees both weak and strong challengers challenge and the defender submit with certainty. A strong reputation undoubtedly benefits the challenger. A weak reputation also has its effects. In the game of Figure 5.05,
the defender's initial belief that the challenger is strong has fallen to 0.1 and the result is a minuscule probability $p = 0.0555556$ that the weak challenger will indeed challenge.

The solution of Figure 5.04 is called a "pooling" equilibrium whereas those of Figures 5.03 and 5.05 are called "semi-separating". There is a third kind of equilibrium that arises in certain structures. Intuitively, a very high belief that the challenger is weak should convince the defender to fight and deter the weak challenger to challenge. Surprisingly, this cannot be the case here (prove it). In order for this situation to occur, one needs other assumptions. For instance, a weak defender who has no stomach for confrontation may have the following payoffs and yield the solution:
Figure 5.06 shows a "separating" equilibrium. The defender is able to determine the challenger's type through his rational behavior.

In many instances, the uncertainty is reversed. Instead of the defender being unsure of the challenger's type it is the challenger who is unsure. But since the challenger has the first move, the game structure must be a little more complex:
Note the reversal of the roles: the upper part no longer means a strong challenger. Instead it means a weak defender. There are two (PBE) solutions: in one case the challenger is deterred from initiating a crisis, the defender is expected to resist with certainty regardless of his type and the challenger would back down with updated beliefs that the defender is strong with certainty. In the second PBE, the weak defender resists only with probability $p = 0.277778$ while the strong defender resists with certainty. The challenger then escalates with probability $p = 0.75$.

The game structure of Figure 5.07 is called a "screening" game. The challenger screens the defender for his type through his rational behavior. Of course, this allows the weak to pass for a strong when the equilibrium is semi-separating.

Although initial beliefs are critical in determining that rational unfolding of the game, the theory is often silent about where they come from. In many applications, these beliefs are an expression of prior history although the history doesn't explicitly appear in the game and it could precisely result from an incentive to establish the right reputation. Consider a game where the defender faces two potential challengers, one after the other:

![Figure 5.08](image)

In Figure 5.08 we assume that the defender can be challenged by a second player should the first one do so initially. In the solution of Figure 5.09 the defender resists with certainty when weak and with probability when strong! But this is because the payoffs reflect an incentive to the trap the two challengers into a three-way struggle from which the strong defender expects to come out a strong winner.
Solving games of incomplete information analytically is not unlike solving other games on a tree: one proceeds by backward induction. The main objective of the analysis is to describe all possible PBEs depending on the initial beliefs. I illustrate the procedure on the game of Figure 5.02 with $p$ denoting the chance probability (initial belief) that Challenger is strong.

There are three cases for the defender's choices at his (red) information set:
(a) Submit is better than fight;
(b) Fight is better than submit; and
(c) The two choices yield the same expected payoffs.

These three cases depend on the (updated) beliefs at nodes D1 and D2. Let $b$ denote the defender's belief to be at node D1 (i.e. that the challenger is strong). Further let $E_{fgt}$ and $E_{sub}$ denote the defender's expected payoffs of the choices fight ($fgt$) and submit ($sub$) respectively. One easily obtains:

$$E_{sub} = -1$$
$$E_{fgt} = -2b + 1 - b = 1 - 3b$$

Case (a) therefore requires $-1 > 1 - 3b$ or $b > b^* = \frac{2}{3}$, case (b) needs $b < b^*$ and case (c) needs $b = b^*$. We discuss these cases in turns:

(a) Since submit is better than fight, the challenger expects a utility of 1 at each of the two nodes D1 and D2. Since this is strictly better than stay at both nodes C1 and C2,
the challenger will challenge in certainty. The updated beliefs should therefore exactly match the initial beliefs, and this requires that $p = b > b^\ast$. In that case we have a pooling PBE. This is what happens in Figure 5.04.

(b) Since fight is better than submit, the challenger expects a utility of 2 at D1 and $-1$ at D2. Therefore he should challenge at C1 and stay at C2. The updated belief should therefore be given by $b = 1$. This is incompatible with case (b) that requires $b < b^\ast$. This situation cannot occur in a PBE.

(c) Since fight and submit are equally attractive for the defender, he may use these choices with probability, say $q$ for fight ($1 - q$ for submit). The challenger’s resulting expected payoffs at D1 and D2 are then $E_{D1} = 1 + q$ and $E_{D2} = 1 - 2q$. Clearly, $E_{D1}$ is greater than the payoff of stay at C1 and a strong challenger will therefore challenge for any value of $q$. But $E_{D2}$ is the same as the payoff of stay at C2 when $q = \frac{1}{2}$. In that case, the challenge may challenge with some probability $x$ at C2. If instead $q < \frac{1}{2}$ then the challenger will challenge and we are back to case (a) and if $q > \frac{1}{2}$ then the challenger will stay at C2 and we are back to case (b).

With probability $x$ of challenge at C2 we obtain the updated beliefs $b = \frac{p}{p + x(1-p)}$. This must match exactly $b^\ast = \frac{2}{3}$ in order to achieve a PBE. This yields $x = \frac{p}{2(1-p)}$ which is a probability provided $p < b^\ast = \frac{2}{3}$. This PBE is then semi-separating as in Figure 5.05. Clearly, there are no other PBEs in this game.

**Homework**

1. The game of Figure 5.10 is known as Selten's Horse.
   (a) Using GamePlan find all it perfect equilibria.
   (b) Using a formal analysis reconstruct all these equilibria.

2. Construct all equilibria of the game of Figure 5.07 for arbitrary initial beliefs using formal analysis.